NEHRU COLLEGE OF ENGINEERING AND RESEARCH CENTRE
(NAAC Accredited)
(Approved by AICTE, Affiliated to APJ Abdul Kalam Technological University, Kerala)

DEPARTMENT OF MECHANICAL ENGINEERING

COURSE MATERIALS

MAT 101 LINEAR ALGEBRA AND CALCULUS

VISION OF THE INSTITUTION

To mould true citizens who are millennium leaders and catalysts of change through excellence in education.

MISSION OF THE INSTITUTION

NCERC is committed to transform itself into a center of excellence in Learning and Research in Engineering and Frontier Technology and to impart quality education to mould technically competent citizens with moral integrity, social commitment and ethical values.

We intend to facilitate our students to assimilate the latest technological know-how and to imbibe discipline, culture and spiritually, and to mould them in to technological giants, dedicated research scientists and intellectual leaders of the country who can spread the beams of light and happiness among the poor and the underprivileged.

ABOUT DEPARTMENT

- Established in: 2002
- Course offered : B.Tech in Mechanical Engineering

  M.Tech in Machine Design
- Approved by AICTE New Delhi and Accredited by NAAC
- Affiliated to the University of Dr. A P J Abdul Kalam Technological University.

DEPARTMENT VISION

Producing internationally competitive Mechanical Engineers with social responsibilities and sustainable employability through viable strategies as well as competent exposure oriented quality education.

DEPARTMENT MISSION

M1	Imparting high impact education by providing conductive teaching learning environment.
M2	Fostering effective modes of continuous learning process with moral and ethical values.
M3	Enhancing leadership qualities with social commitment, professional attitude, unity, team spirit and communication skill.
M4	Introducing present scenario in research and development through collaborative efforts blended with industry and institution.

PROGRAMME EDUCATIONAL OBJECTIVES

PEO No.	Program Educational Objectives Statements
PEO1	Graduates shall have strong practical and technical exposures in the field of Mechanical Engineering and will contribute to the society through Innovation and Enterprise.
PEO2	Graduates will have the demonstrated ability to analyze, formulate and solve design engineering/thermal engineering/materials and manufacturing/design issues and real life problems.
PEO3	Graduates will be capable of pursuing Mechanical Engineering profession with good communication skills, leadership qualities, team spirit and professional ethics.
PEO4	Graduates will sustain an appetite for continuous learning by pursuing higher education and research in the allied areas of technology.

PROGRAM OUTCOMES (POS)

Engineering Graduates will be able to:

1. **Engineering knowledge**: Apply the knowledge of mathematics, science, engineering fundamentals, and an engineering specialization to the solution of complex engineering problems.
2. **Problem analysis**: Identify, formulate, review research literature, and analyze complex engineering problems reaching substantiated conclusions using first principles of mathematics, natural sciences, and engineering sciences.
3. **Design/development of solutions**: Design solutions for complex engineering problems and design system components or processes that meet the specified needs with appropriate consideration for the public health and safety, and the cultural, societal, and environmental considerations.
4. **Conduct investigations of complex problems**: Use research-based knowledge and research methods including design of experiments, analysis and interpretation of data, and synthesis of the information to provide valid conclusions.
5. **Modern tool usage**: Create, select, and apply appropriate techniques, resources, and modern engineering and IT tools including prediction and modeling to complex engineering activities with an understanding of the limitations.
6. **The engineer and society**: Apply reasoning informed by the contextual knowledge to assess societal, health, safety, legal and cultural issues and the consequent responsibilities relevant to the professional engineering practice.
7. **Environment and sustainability**: Understand the impact of the professional engineering solutions in societal and environmental contexts, and demonstrate the knowledge of, and need for sustainable development.
8. **Ethics**: Apply ethical principles and commit to professional ethics and responsibilities and norms of the engineering practice.
9. **Individual and team work**: Function effectively as an individual, and as a member or leader in diverse teams, and in multidisciplinary settings.
10. **Communication**: Communicate effectively on complex engineering activities with the engineering community and with society at large, such as, being able to comprehend and write effective reports and design documentation, make effective presentations, and give and receive clear instructions.
11. **Project management and finance**: Demonstrate knowledge and understanding of the engineering and management principles and apply these to one’s own work, as a member and leader in a team, to manage projects and in multidisciplinary environments.
12. **Life-long learning**: Recognize the need for, and have the preparation and ability to engage in independent and life-long learning in the broadest context of technological change.

PROGRAM SPECIFIC OUTCOMES (PSO)

PSO1	Graduates able to apply principles of engineering, basic sciences and analytics including multi-variant calculus and higher order partial differential equations.
PSO2	Graduates able to perform modelling, analysing, designing and simulating physical systems, components and processes.
PSO3	Graduates able to work professionally on mechanical systems, thermal systems and production systems.

COURSE OUTCOMES

CO1	solve systems of linear equations, diagonalize matrices and characterise quadratic forms
CO2	compute the partial and total derivatives and maxima and minima of multivariable functions
CO3	compute multiple integrals and apply them to find areas and volumes of geometrical shapes, mass and centre of gravity of plane laminas
CO4	perform various tests to determine whether a given series is convergent, absolutely convergent or conditionally convergent
CO5	determine the Taylor and Fourier series expansion of functions and learn their applications.

MAPPING OF COURSE OUTCOMES WITH PROGRAM OUTCOMES

	PO 1	PO 2	PO 3	PO 4	PO 5	PO 6	PO 7	PO 8	PO 9	PO 10	PO 11	PO 12
CO 1	**3**	**3**	**3**	**3**	**2**	**1**			**1**	**2**		**2**
CO2	**3**	**3**	**3**	**3**	**2**	**1**			**1**	**2**		**2**
CO3	**3**	**3**	**3**	**3**	**2**	**1**			**1**	**2**		**2**
CO4	**3**	**2**	**3**	**2**	**1**	**1**			**1**	**2**		**2**
CO5	**3**	**3**	**3**	**3**	**2**	**1**			**1**	**2**		**2**

	PSO1	PSO2	PSO3
CO1	1	1	
CO2	2	1	
CO3	2	1	
CO4	1	1	
CO5	1	1	

Note: H-Highly correlated=3, M-Medium correlated=2, L-Less correlated=1

SYLLABUS

Module 1 (Linear algebra)

(Text 2: Relevant topics from sections 7.3, 7.4, 7.5, 8.1,8.3,8.4)

Systems of linear equations, Solution by Gauss elimination, row echelon form and rank of a matrix, fundamental theorem for linear systems (homogeneous and non-homogeneous, without proof), Eigen values and eigen vectors. Diagonaliztion of matrices, orthogonal transformation, quadratic forms and their canonical forms.

Module 2 (multivariable calculus-Differentiation)

(Text 1: Relevant topics from sections 13.3, 13.4, 13.5, 13.8)

Concept of limit and continuity of functions of two variables, partial derivatives, Differentials, Local Linear approximations, chain rule, total derivative, Relative maxima and minima, Absolute maxima and minima on closed and bounded set.

Module 3(multivariable calculus-Integration)

(Text 1: Relevant topics from sections 14.1, 14.2, 14.3, 14.5, 14.6, 14.8)

Double integrals (Cartesian), reversing the order of integration, Change of coordinates (Cartesian to polar), finding areas and volume using double integrals, mass and centre of gravity of inhomogeneous laminas using double integral. Triple integrals, volume calculated as triple integral, triple integral in cylindrical and spherical coordinates (computations involving spheres, cylinders).

Module 4 (sequences and series)

(Text 1: Relevant topics from sections 9.1, 9.3, 9.4, 9.5, 9.6)

Convergence of sequences and series, convergence of geometric series and p-series(without proof), test of convergence (comparison, ratio and root tests without proof); Alternating series and Leibnitz test, absolute and conditional convergence.

Module 5 (Series representation of functions)

(Text 1: Relevant topics from sections 9.8, 9.9. Text 2: Relevant topics from sections 11.1, 11.2, 11.6)

Taylor series (without proof, assuming the possibility of power series expansion in appropriate domains), Binomial series and series representation of exponential, trigonometric, logarithmic functions (without proofs of convergence); Fourier series, Euler formulas, Convergence of Fourier series (without proof), half range sine and cosine series, Parseval's theorem (without proof).

Text Books

1. H. Anton, I. Biven,S.Davis, "Calculus", Wiley, 10th edition, 2015.
2. Erwin Kreyszig, Advanced Engineering Mathematics, 10thEdition, John Wiley & Sons, 2016.

Reference Books

1. J. Stewart, Essential Calculus, Cengage, 2nd edition, 2017
2. G.B. Thomas and R.L. Finney, Calculus and Analytic geometry, 9 th Edition, Pearson, Reprint, 2002.
3. Peter V. O'Neil, Advanced Engineering Mathematics , Cengage, 7th Edition, 2012
4. Veerarajan T., Engineering Mathematics for first year, Tata McGraw-Hill, New Delhi, 2008.
5. B.S. Grewal, Higher Engineering Mathematics, Khanna Publishers, 36 Edition, 2010.

Course Contents and Lecture Schedule

No	Topic	No. of Lectures
1	**Linear Algebra (10 hours)**	
1.1	Systems of linear equations, Solution by Gauss elimination	1
1.2	Row echelon form, finding rank from row echelon form, fundamental theorem for linear systems	3
1.3	Eigen values and eigen vectors	2
1.4	Diagonaliztion of matrices, orthogonal transformation, quadratic forms	4
2	**Multivariable calculus-Differentiation (8 hours)**	
2.1	Concept of limit and continuity of functions of two variables, partial derivatives	2
2.2	Differentials, Local Linear approximations	2
2.3	Chain rule, total derivative	2
2.4	Maxima and minima	2

3	**Multivariable calculus-Integration (10 hours)**	
3.1	Double integrals (Cartesian)-evaluation	2
3.2	Change of order of integration in double integrals, change of coordinates (Cartesian to polar),	2
3.3	Finding areas and volumes, mass and centre of gravity of plane laminas	3
3.4	Triple integrals	3

4	**Sequences and series (8 hours)**	
4.1	Convergence of sequences and series, geometric and p-series	2
4.2	Test of convergence(comparison, ratio and root)	4
4.3	Alternating series and Leibnitz test, absolute and conditional convergence	2

5	**Series representation of functions (9 hours)**	
5.1	Taylor series, Binomial series and series representation of exponential, trigonometric, logarithmic functions;	3
5.2	Fourier series, Euler formulas, Convergence of Fourier series(Dirichlet's conditions)	3
5.3	Half range sine and cosine series, Parseval's theorem.	3

QUESTION BANK

MODULE I

Q:NO:	QUESTIONS	CO	KL	PAGE NO:
1	Solve the linear system whose augmented matrix is $\begin{bmatrix} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0.6 & 1.5 & 1.5 & -5.4 & 2.7 \\ 1.2 & -0.3 & -0.3 & 2.4 & 2.1 \end{bmatrix}$	CO1	K3	14
2	Solve the linear system $10x + 4y - 2z = 14$ $-3w - 15x + y + 2z = 0$ $w + x + y = 6$ $8w - 5x + 5y - 10z = 26$	CO1	K3	15
3	Check for consistence of the system $x + y + z = 1, \quad x + 2y + 4z = 2, \; x + 4y + 10z = 4$	CO1	K2	16
4	Show that the equations $3x + 4y + 5z = a, \; 4x + 5y + 6z = b, \quad 5x + 6y + 7z = c$ do not have a solution unless $a + c = 2b$	CO1	K2	18
5	Find the value of β if the system has a non-trivial solution $x_1 + x_2 = 0, \; x_2 + x_3 = 0$ $x_1 + x_2 + \beta x_3 = 0.$	CO1	K1	17
6	Solve the following by Gauss elimination $y + z - 2w = 0$, $2x - 2y - 3z + 6w = 2, 4x + y + z - 2w = 4.$	CO1	K3	15
7	Is the matrix A is orthogonal if $A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$	CO1	K2	28
8	Find an eigen basis and Diagonalize the matrix $A = \begin{bmatrix} -5 & -6 & 6 \\ -9 & -8 & 12 \\ -12 & -12 & 16 \end{bmatrix}$	CO1	K3	35
9	Find the rank . $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -4 \\ 0 & 4 & 0 \end{pmatrix}$	CO1	K2	20
10	Find out what type of conic section does follows quadratic form represents and transform it into principal axes if $Q = 4x_1^2 + 24x_1x_2 - 14x_2^2 = 20$	CO1	K1	42
11	Diagonalise $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$	CO1	K3	40

MODULE II

1	Given f=$e^x \sin y$ show that the function satisfies the Laplace equation $f_{xx} + f_{yy} = 0$	CO2	K2	46
2	If $f(x,y) = x^2y^3 + x^4y$ find f_{xy}	CO2	K1	45
3	Compute the differential dz of the function $z = \tan^{-1}(xy)$	CO2	K2	47
4	Find the slope of the surface $z = \sqrt{3x+2y}$ in the $y-$direction at the point $(4,2)$	CO2	K2	46
5	Find the derivative of $w = x^2 + y^2$ with respect to 't' along the path $x = at^2, y = 2at$	CO2	K2	45
6	Given $z = e^{xy}$ $x = 2u + v, y = \frac{v}{u}$ find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$	CO2	K1	47
7	Use chain rule find $\frac{dw}{ds}$ at $s = \frac{1}{4}$ if $w = r^2$-r $\tan\theta$, $r = \sqrt{s}$, $\theta = \pi s$	CO2	K3	53
8	If $u = \log(x^3 + y^3 + z^3 - 3xyz)$ show that $(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z})^2 u = \frac{-9}{(x+y+z)^2}$	CO2	K4	54
9	If $u = \frac{x^3+y^3}{x-y}$ Find $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}$	CO2	K1	49
10	If $w = 3xy^2z^3$, $y = 3x^2 + 2$, $z = \sqrt{x-1}$ find $\frac{dw}{dx}$ and $\frac{dw}{dy}$	CO2	K1	48
11	Locate all relative maxima, relative minima and saddle point if any of $f(x,y) = y^2 + xy + 4y + 2x + 3$	CO2	K3	59
12	Let $L(x,y)$ denote the local linear approximation to $f(x,y) = \sqrt{x^2+y^2}$ at the point $(3,4)$. Compare the error in approximating $f(3.04, 3.98) = \sqrt{(3.04)^2 + (3.98)^2}$ by $L(3.04, 3.98)$ with the distance between the points (3,4) and (3.02,3.98)	CO2	K3	61
13	Find the absolute extrema of the function $f(x,y) = xy - 4x$ of R where R is the triangular region with the vertices (0,0), (0,4) and (4,0)	CO2	K3	62
14	If u=f($\frac{x}{y}, \frac{y}{z}, \frac{z}{x}$) prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$	CO2	K2	53

MODULE III

1	Evaluate $\int_1^a \int_1^b \frac{dydx}{xy}$	CO3	K1	64
2	The line $y = 2 - x$ and the parabola $y = x^2$ intersects at the points (-2,4) and (1,1). If R is the region enclosed by $y = 2 - x$ and $y = x^2$ then find $\iint_R y\, dA$	CO3	K3	68
3	Find the area bounded by the $x-axis$, $y = 2x$ and $x + y = 1$ using double integration	CO3	K3	69
4	Sketch the region of integration and evaluate the integral $\int_1^2 \int_y^{y^2} dxdy$ by changing the order of integration.	CO3	K3	72
5	Sketch the region of integration and evaluate $\int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2) dydx$	CO3	K3	70
6	By changing the order of integration evaluate $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\ dydx$	CO3	K4	75
7	Evaluate $\int_0^1 \int_0^1 \frac{dxdy}{\sqrt{1-x^2}\sqrt{1-y^2}}$	CO3	K1	66
8	Evaluate $\iint_R \frac{\sin x}{x}\, dA$ where R is the triangular region bounded by $x-axis$, $y = x$, and $x = 1$	CO3	K2	67
9	Change the order of integration and evaluate $\int_0^1 \int_x^1 \frac{x}{x^2+y^2}\ dxdy$	CO3	K4	77
10	Find the area bounded by the parabolas $y^2 = 4x$ and $x^2 = -\frac{y}{2}$	CO3	K2	66
11	Evaluate $\iint_R \mathbf{x^2\, dA}$ over the region **R** enclosed between $\mathbf{y = \frac{16}{x}}$, $\mathbf{y = x}$, **and** $\mathbf{x = 8}$	CO3	K3	72
12	Evaluate $\int_0^3 \int_0^2 \int_0^1 (xyz) dxdydz$	CO3	K1	65
13	find the volume bounded by the cylinder $x^2 + y^2 = 4$ the planes z=0 and y+z=3	CO3	K2	79
14	Use a triple integral to find the volume of the solid within the cylinder $y = x^2$ and the plane $y + z = 4$, $z = 0$	CO3	K3	80

MODULE IV

1	Show that the series $\sum_{k=1}^{\infty} \frac{\cos k}{k^2}$ is convergent	CO4	K3	87
2	Test the convergence of $\frac{x}{1.2} + \frac{x^2}{2.3} + \frac{x^3}{3.4} + \cdots \ldots \ldots$	CO4	K2	85
3	Check whether the series $\sum_{k=1}^{\infty} \frac{1}{2k-1}$ converges or not	CO4	K2	86
4	Determine whether the series $\sum_{k=1}^{\infty} (\frac{3}{4})^{k+2}$ converges and if so find its sum	CO4	K1	85
5	Test the nature of the series $\sum_{k=1}^{\infty} \frac{4k^3-6k+5}{8k^7+k-8}$	CO4	K3	89
6	Check whether the series $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$ converges or not	CO4	K3	92
7	Test the convergence of $\sum_{n=1}^{\infty} (\frac{n}{n+1})^{n^2}$	CO4	K3	96
8	Check whether the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1} k^k}{k!}$ Absolutely convergent or not	CO4	K4	102
9	Determine whether the alternating series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}\ k+1}{k(k+4)}$ is absolutely convergent	CO4	K4	103
10	Show that the series $\sum_{k=1}^{\infty} (\frac{1}{2})^k$ converges and $\sum_{k=1}^{\infty} (-1)^k$ diverges	CO4	K3	105
11	Check whether the series $\sum_{n=1}^{\infty} \frac{(-1)^n (2n-1)!}{3^n}$ Absolutely convergent or not	CO4	K4	104
12	Examine the convergence of the series $\sum_{k=1}^{\infty} \frac{k^k}{k!}$	CO4	K3	99
13	Use ratio test for absolute convergence to find whether the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1} 2^k}{k!}$ Converges	CO4	K3	109
14	Determine whether the series $\sum_{k=1}^{\infty} \frac{5}{4^k}$ converges. If so find sum	CO4	K2	86

MODULE V

1	Find the Fourier series expansion of $f(x) = e^{-x}$ $in - c < x < c$	CO5	K3	123
2	Obtain the Fourier series for the function $f(x) = \begin{cases} 1+\frac{2x}{\pi} & -\pi \le x \le 0 \\ 1-\frac{2x}{\pi} & 0 \le x \le \pi \end{cases}$	CO5	K3	125
3	Develop the Fourier series of $f(x) = x^2 \quad -2 < x < 2$	CO5	K4	127
4	Develop the Fourier series of $f(x) = e^{-x} \quad -l < x < l$	CO5	K2	129
5	Obtain the Fourier series for the function $f(x) = \begin{cases} -1+x & -\pi \le x \le 0 \\ 1+x & 0 \le x \le \pi \end{cases}$	CO5	K3	130
6	Develop the Fourier sine series of $f(x) = \begin{cases} x & 0 < x < 2 \\ 4-x & 2 < x < 4 \end{cases}$	CO5	K3	135
7	Find the Maclaurin's series for $\frac{1}{1-x}$	CO5	K1	121
8	Find Maclaurin series for the function xe^x	CO5	K1	120
9	.Find the Taylor series expansion of $\log\cos x$ about the point $x = \frac{\pi}{3}$	CO5	K2	119
10	.Find the Taylor series of $\frac{1}{x+2}$ about $x = 1$	CO5	K2	118
11	Find the Fourier series of the periodic function f(x) of period 4, where f(x)= $\begin{matrix} 2 & -2 \le x \le 0 \\ x & 0 \le x \le 2 \end{matrix}$ Deduce that (i) $1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots \ldots = \frac{\pi^2}{8}$	CO5	K4	137
12	Find the Taylor series of $\frac{1}{x}$ about $x = 1$	CO5	K2	119
13	Find the half range sine series for the function $f(x) = \begin{cases} x & 0 < x < 1 \\ 2-x & 1 < x < 2 \end{cases}$	CO5	K3	140

Linear Systems of Equations

A linear system of m equations in n unknowns $x_1, x_2, \ldots x_n$ is a set of equations of the form

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{array}\right\} \longrightarrow (1)$$

The system is called linear because each variable x_j appears in the first power only, just as in the equation of a straight line, $a_{11}, a_{12}, \ldots a_{mn}$ are given numbers, called coefficients of the system. $b_1, b_2, \ldots b_m$ on the right are also given numbers. If all the b_j are zero, then (1) is called a homogeneous system. If atleast one b_j is not zero, then (1) is called non homogeneous system.

A Solution of (1) is a set of numbers $x_1, x_2, \ldots x_n$ that satisfies all the m equations.

A Solution vector of (1) is a vector X whose components form a solution of (1). If the system (1) is homogeneous, it always has the trivial solution $x_1=0, x_2=0 \ldots x_n=0$.

Matrix form of the Linear System

$Ax = b$ is the matrix of the linear system of equations.

Where $A = [a_{jk}]_{m \times n}$ is the coefficient matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$[A|b] \text{ or } \tilde{A} = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$ is called

<u>Augumented Matrix</u> of the System (1).

<u>Gauss Elimination and Back Substitution</u>

In this method. We reduce the augumented matrix corresponding linear system into upper triangular form. Then we apply the back Substitution Method.

<u>Elementary Row operations for matrices</u>

* Interchange of two rows
* Addition of a constant multiple of one row to another row.
* Multiplication of a row by a nonzero constant c

Note: Row equivalent linear system have the same set of solutions.

Gauss Elimination: Three possible cases

At the end of the Gauss elimination the form of the coefficient matrix, the augmented matrix and the system itself are called row echelon form. The number of nonzero rows in the row reduced coefficient matrix A is called Rank of A.

There are three cases.

1) No Solution

If Rank of [A] $\neq$ Rank of [A|B] then the system is inconsistent and has no solution

2) Unique Solution

If Rank of [A] = Rank of [A B] = n, no. of unknowns then system is consistent and unique solution.

3) Infinitely many solutions

If Rank of [A] = Rank of [A B] < n, no. of unknowns then system is consistent and infinitely many solutions.

If Rank of [A] = r then choose arbitrarily values for n−r variables and solve remaining.

Pbms.

1 Solve the following system of equations

$x+y+z=8 \qquad x-y+2z=6 \qquad 3x+5y-7z=14$

Augmented matrix
$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 8 \\ 1 & -1 & 2 & 6 \\ 3 & 5 & -7 & 14 \end{array}\right]$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - 3R_1$
$$\left[\begin{array}{cccc} 1 & 1 & 1 & 8 \\ 0 & -2 & 1 & -2 \\ 0 & 2 & -10 & -10 \end{array}\right]$$

$R_3 \to R_3 + R_2$
$$\left[\begin{array}{cccc} 1 & 1 & 1 & 8 \\ 0 & -2 & 1 & -2 \\ 0 & 0 & -9 & -12 \end{array}\right]$$

Here Rank[A] = Rank[AB] = No of unknowns

$\Rightarrow$ Consistent & unique soln

$-9z=-12 \qquad z = 4/3 \qquad -2y+z=-2$

$-2y+4/3=-2 \qquad y=5/3$

$x+y+z=8 \qquad x+5/3+4/3=8 \qquad x=5$

$x=5 \qquad y=5/3 \qquad z=4/3$

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2. $-2b+3c=1 \qquad 3a+6b-3c=-2 \qquad 6a+6b+3c=5$

Augmented Matrix
$$\left[\begin{array}{cccc} 0 & -2 & 3 & 1 \\ 3 & 6 & -3 & -2 \\ 6 & 6 & 3 & 5 \end{array}\right]$$

$R_2 \leftrightarrow R_1$
$$\left[\begin{array}{cccc} 3 & 6 & -3 & -2 \\ 0 & -2 & 3 & 1 \\ 6 & 6 & 3 & 5 \end{array}\right]$$

$R_3 \to R_3 - 2R_1.$
$$\left[\begin{array}{cccc} 3 & 6 & -3 & -2 \\ 0 & -2 & 3 & 1 \\ 0 & -6 & 9 & 9 \end{array}\right]$$

$R_3 \to R_3 - 3R_2$
$$\left[\begin{array}{cccc} 3 & 6 & -3 & -2 \\ 0 & -2 & 3 & 1 \\ 0 & 0 & 0 & 6 \end{array}\right]$$

Rank [A] = 2 $\neq$ Rank of [AB] = 3

The system inconsistent and no solutions.

3) $x+y+z=1$

$x+2y+3z=4$

$x+3y+5z=7$

$x+4y+7z=10$

Augumented Matrix
$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 7 \\ 1 & 4 & 7 & 10 \end{array}\right]$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$
$R_4 \to R_4 - R_1$
$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 6 \\ 0 & 3 & 6 & 9 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$
$R_4 \to R_4 - 3R_2$
$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank of $[A]$ = Rank of $[AB] = 2$ Consistant

Rank < 3 Many solns.

put $z=t$

$y+2z=3$

$y+2t=3$ $\quad y=3-2t$

$x+y+z=1$ $\quad x+(3-2t)+t=1$

$x = 1-3+2t-t$

$= \underline{\underline{-2+t}}$

$x=-2+t$ $\quad y=3-2t$ $\quad z=t$

Pbms

1 Solve the linear system given explicitly or by its augmented matrix. Show details.

$x_1 - x_2 + x_3 = 0$, $-x_1 + x_2 - x_3 = 0$, $10x_2 + 25x_3 = 90$,

$20x_1 + 10x_2 = 80$

A: Augmented matrix

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{array}\right]$$

Call the first row of A the pivot row and the first eqn the pivot eqn its first element is called pivot element.

$R_2 \to R_2 + R_1$.

$R_4 \to R_4 - 20R_1$.

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \end{bmatrix}$$

Here new second eqn is the pivot eqn. But since it has no x_2-terms we must change order of eqn. Move the second eqn to last and third and fourth eqn one place up. This is called partial pivoting.

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & \boxed{10}^{\text{pivot}} & 25 & 90 \\ 0 & 30 & -20 & 80 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 \to R_3 - 3R_2$

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 10 & -20 & 80 \\ 0 & 0 & -95 & -190 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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Back Substitution

$$-95x_3 = -190 \qquad x_3 = \frac{190}{95} = 2 \qquad \boxed{x_3 = 2}$$

$$10x_2 + 25x_3 = 90 \qquad 10x_2 = 90 - 50 = 40 \qquad \boxed{x_2 = 4}$$

$$x_2 = 4$$

$$x_1 - x_2 + x_3 = 0 \qquad x_1 - 4 + 2 = 0 \quad x_1 = 2 \qquad \boxed{x_1 = 2}$$

$$x_1 = 2, \quad x_2 = 4, \quad x_3 = 2.$$

---

2. $-3x + 8y = 5$ Augmented Matrix $[A|B] = \begin{bmatrix} -3 & 8 & 5 \\ 8 & -12 & -11 \end{bmatrix}$

$8x - 12y = -11$

$$R_2 \to R_2 - \frac{8}{-3}R_1 \qquad \begin{bmatrix} -3 & 8 & 5 \\ 0 & \frac{28}{3} & 7/3 \end{bmatrix}$$

$$\frac{28}{3}y = 7/3 \qquad y = 7/3 \times \frac{3}{28} = 1/4.$$

$$-3x + 8y = 5 \implies -3x + 8 \times 1/4 = 5$$

$$-3x = +3 \qquad x = -1$$

$$x = -1 \qquad y = 1/4$$

---

3 $8y + 6z = -4, \quad -2x + 4y - 6z = 18, \quad x + y - z = 2.$

Augmented Matrix
$$\begin{bmatrix} 0 & 8 & 6 & -4 \\ -2 & 4 & -6 & 18 \\ 1 & 1 & -1 & 2 \end{bmatrix}$$

$R_2 \leftrightarrow R_1$.
$$\begin{bmatrix} -2 & 4 & -6 & 18 \\ 0 & 8 & 6 & -4 \\ 1 & 1 & -1 & 2 \end{bmatrix}$$

$R_3 \to R_3 - \frac{1}{-2}R_1$.
$$\begin{bmatrix} -2 & 4 & -6 & 18 \\ 0 & 8 & 6 & -4 \\ 0 & 3 & -4 & 11 \end{bmatrix}$$

$R_3 \to R_3 - \frac{3}{8}R_2$
$$\begin{bmatrix} -2 & 4 & -6 & 18 \\ 0 & 8 & 6 & -4 \\ 0 & 0 & \frac{-25}{4} & \frac{25}{2} \end{bmatrix}$$

$$-\frac{25}{4}z = \frac{25}{2} \implies z = -2. \qquad 8y + 6z = -4 \implies y = 1 \qquad -2x + 4y - 6z = 18 \implies x = -1$$

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$x = -1 \quad y = 1 \quad z = -2.$

4

$$\begin{bmatrix} 13 & 12 & 6 \\ -4 & 7 & 73 \\ 4 & 5 & 11 \end{bmatrix}$$

$R_2 \to R_2 - \frac{-4}{13} R_1$

$R_3 \to R_3 - \frac{4}{13} R_1$

$$\begin{bmatrix} 13 & 12 & 16. \\ 0 & \frac{139}{13} & \frac{973}{13} \\ 0 & \frac{17}{13} & \frac{119}{13} \end{bmatrix}$$

$R_3 \to R_3 - \frac{17}{139} R_2$

$$\begin{bmatrix} 13 & 12 & 16 \\ 0 & \frac{139}{13} & \frac{973}{13} \\ 0 & 0 & 0 \end{bmatrix}$$

$\frac{139}{13} x_2 = \frac{973}{13} \qquad x_2 = \frac{973}{139} = 7$

$13x_1 + 12x_2 = 6. \qquad x_1 = -6$

$$\boxed{\begin{matrix} x_1 = -6 \\ x_2 = 7 \end{matrix}}$$

5 Solve.

$$\left[\begin{array}{cccc|c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0.6 & 1.5 & 1.5 & -5.4 & 2.7 \\ 1.2 & -0.3 & -0.3 & 2.4 & 2.1 \end{array}\right]$$

$R_2 \to R_2 - \frac{0.6}{3.0} R_1 \Rightarrow R_2 \to R_2 - .2R_1.$

$R_3 \to R_3 - \frac{1.2}{3.0} R_1 \Rightarrow R_3 \to R_3 - .4R_1$

$$\begin{bmatrix} 3 & 2 & 2 & -5 & 8 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & -1.1 & -1.1 & 4.4 & -1.1 \end{bmatrix}$$

$R_3 \to R_3 - - R_2 \Rightarrow R_3 \to R_3 + R_2$

$$\begin{bmatrix} 3 & 2 & 2 & -5 & 8 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Here Rank of [A] = Rank of [A B] = 2 Consistent.

Rank < No of unknowns = 4 $\Rightarrow$ Infinite Solutions

Choose [4-2 = 2 Variables arbitrarily)

Put $x_3 = s$, $x_4 = t$

$1.1x_2 + 1.1x_3 - 4.4x_4 = 1.1 \Rightarrow \underline{x_2 = 1 - s + 4t}$

$3x_1 + 2x_2 + 2x_3 - 5x_4 = 8 \Rightarrow x_1 = 2 - t$

$x_1 = 2 - t \quad x_2 = 1 - s + 4t \quad x_3 = s \quad x_4 = t$

---

6 Solve. $\left[\begin{array}{ccc:c} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{array}\right]$ No of unknowns = 3.

$R_2 \to R_2 - 2/3 R_1$

$R_3 \to R_3 - 6/3 R_2$

$R_3 - 2R_2$.

$\begin{bmatrix} 3 & 2 & 1 & 3 \\ 0 & -1/3 & 1/3 & -2 \\ 0 & 0 & 0 & 12 \end{bmatrix}$

Rank of A (=3) ≠ Rank of [A B] (=4)

In Consistent, No solution.

---

7 $\begin{bmatrix} 4 & 0 & 6 \\ -1 & 1 & -1 \\ 2 & -4 & 1 \end{bmatrix}$ No of unknowns = 2

$R_2 \to R_2 - {-1/4} R_1$

$R_3 \to R_3 - \frac{1}{2} R_1$

$\begin{bmatrix} 4 & 0 & 6 \\ 0 & 1 & .5 \\ 0 & -4 & -2 \end{bmatrix}$

$R_3 \to R_3 - {-4}R_2$

$R_3 + 4R_2$

$\begin{bmatrix} 4 & 0 & 6 \\ 0 & 1 & .5 \\ 0 & 0 & 0 \end{bmatrix}$

$y = .5 \quad 4x = 6 \quad x = 3/2$.

$\boxed{\begin{array}{l} y = 1/2 \\ x = 3/2 \end{array}}$

8 $-2y-2z=8$

$3x+4y-5z=8$

Augumented Matrix $\begin{bmatrix} 0 & -2 & -2 & 8 \\ 3 & 4 & -5 & 8 \end{bmatrix}$

$R_2 \leftrightarrow R_1$ $\begin{bmatrix} 3 & 4 & -5 & 8 \\ 0 & -2 & -2 & 8 \end{bmatrix}$

Rank $(A) = 2 =$ Rank $[A\,B]$ $\rightarrow$ consistent

No of unknowns $= 3$ Rank < no of unknowns. $\rightarrow$ Infinite soln

$[3-2=1]$ put $z=t$

$-2y-2z=8$

$-2y = 8+2t$ $\underline{\underline{y=-4-t}}$

$3x+4y-5z=8$

$3x+4(-4-t)-5t=8$

$3x-16-9t=8$

$3x = 24+9t$ $\underline{\underline{x=8+3t}}$

$$\boxed{\begin{aligned} x &= 8+3t \\ y &= -4-t \\ z &= t \end{aligned}}$$

---

9. $y+z-2w=0$

$2x-3y-3z+6w=2$

$4x+y+z-2w=4$

Augumented Matrix $\begin{bmatrix} 0 & 1 & 1 & -2 & 0 \\ 2 & -3 & -3 & 6 & 2 \\ 4 & 1 & 1 & -2 & 4 \end{bmatrix}$

$R_2 \leftrightarrow R_1$ $\begin{bmatrix} 2 & -3 & -3 & 6 & 2 \\ 0 & 1 & 1 & -2 & 0 \\ 4 & 1 & 1 & -2 & 4 \end{bmatrix}$

$R_3 \rightarrow R_3 - 2R_1$ $\begin{bmatrix} 2 & -3 & -3 & 6 & 2 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 7 & 7 & -14 & 0 \end{bmatrix}$

$R_3 \to R_3 - 7R_2$ $\begin{bmatrix} 2 & -3 & -3 & 6 & 2 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ (6).

Rank of [A] = Rank [A B] = 2 < 4 | no. of unknowns

Consistent and infinite solutions.

$\boxed{4-2=2}$ put $w = t$ $z = s$

$y + z - 2w = 0$. $\underline{y = 2t - s}$

$2x - 3y - 3z + 6w = 2$.

$2x = 2 + 3(2t - s) + 3s - 6t$

$2x = 2$ $\underline{x = 1}$

$\boxed{\begin{matrix} x = 1 \\ y = 2t - s \\ w = t, z = s \end{matrix}}$

Linear independence. Rank of a matrix

Given any set of vectors $a_1, a_2, \ldots a_m$ a linear Combination of these vectors is an expression of the form $c_1 a_1 + a_2 a_2 + \ldots + c_m a_m$. Where $c_1, c_2 \ldots c_m$ are any scalars.

Now consider the equation

$$c_1 a_1 + c_2 a_2 + \ldots + c_m a_m = 0.$$

If all $c_j = 0$ $j = 1, 2, \ldots m$ then we say that $a_1, a_2, \ldots a_m$ are linearly independent. If atleast one of $c_j \neq 0$ then we say $a_1, a_2, \ldots a_m$ are linearly dependent.

Pbms

1. Check independence of a vectors in $\mathbb{R}^3$.

$\{(1,1,1), (-1,0,1), (0,-2,1)\}$

A: Let $a, b, c \in \mathbb{R}$ s. $a(1,1,1) + b(-1,0,1) + c(0,-2,1) = 0$

$\Rightarrow a - b = 0$, $a - 2c = 0$ $a + b + c = 0$ $\Rightarrow a = 0, b = 0, c = 0$

$\Rightarrow$ vectors are linearly independent.

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Rank of a matrix A is the maximum number of linearly independent row vectors of A denoted by rank A.

[Rank of a matrix is the number of nonzero rows]

1) Find the rank.

1) $A = \begin{bmatrix} 0 & 0 & 5 \\ 3 & 5 & 0 \\ 5 & 0 & 0 \end{bmatrix}$

$R_2 \leftrightarrow R_1$ $\begin{bmatrix} 3 & 5 & 0 \\ 0 & 0 & 5 \\ 5 & 0 & 0 \end{bmatrix}$

$R_3 \to R_3 - \frac{5}{3}R_1$. $\begin{bmatrix} 3 & 5 & 0 \\ 0 & 0 & 5 \\ 0 & -\frac{25}{3} & 0 \end{bmatrix}$

$R_2 \leftrightarrow R_3$ $\begin{bmatrix} 3 & 5 & 0 \\ 0 & -\frac{25}{3} & 0 \\ 0 & 0 & 5 \end{bmatrix}$ Rank of $[A] = 3$

---

2) $\begin{bmatrix} 2 & -2 & 1 \\ 0 & 4 & 8 \\ 2 & 0 & 4 \end{bmatrix}$

$R_3 \to R_3 - R_1$ $\begin{bmatrix} 2 & -2 & 1 \\ 0 & 4 & 8 \\ 0 & 2 & 3 \end{bmatrix}$

$R_3 \to R_3 - \frac{1}{2}R_2$ $\begin{bmatrix} 2 & -2 & 1 \\ 0 & 4 & 8 \\ 0 & 0 & -1 \end{bmatrix} \Rightarrow$ Rank = 3.

---

3) $\begin{bmatrix} 6 & 0 & -3 & 0 \\ 0 & -1 & 0 & 5 \\ 2 & 0 & -1 & 0 \end{bmatrix}$

$R_3 \to R_3 - \frac{1}{3}R_1$

$$\begin{bmatrix} 6 & 0 & -3 & 0 \\ 0 & -1 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{7}$$

Rank = 2.

Fundamental theorem for linear systems

a) **Existence:** A linear system of m equations in n unknowns $x_1, x_2, \ldots x_n$

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}\right\} \longrightarrow (1)$$

is consistent, that is has solutions, if and only if the coefficient matrix A and the augmented matrix $\tilde{A}$ have the same rank, Here

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \text{ \& } \tilde{A} = \left[\begin{array}{cccc:c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

b) **Uniqueness**

The system (1) has precisely one solution if and only if this common rank r of A and $\tilde{A}$ equals n

(c). **Infinitely many solutions**

If the common rank r is less than n, the system (1) has infinitely many solutions. All of these

Solutions are obtained by determining r suitable unknowns in terms of the remaining n-r unknowns, to which arbitrarily values can be assigned.

d) Gauss Elimination

If solutions exist, they can all be obtained by the Gauss Elimination.

Eigen Values and Eigen Vectors

Characteristic equation

Let A be an nxn matrix, then the equation $|A-\lambda I|=0$ is called the characteristic equation and its roots are called characteristic roots or latent roots or eigen values of A

Pbms

1 Find the eigen values of the matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}$$

$$|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda)^2 - 1 = 0$$

$$4 - 4\lambda + \lambda^2 - 1 = 0$$

$$\lambda^2 - 4\lambda + 3 = 0$$

$$\lambda = 1, 3$$

Eigen values are 1 & 3.

---

2 Find the eigen values of the matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

$$|A-\lambda I|=0 \Rightarrow \lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

S_1 = Sum of diagonal elements $= 2+3+2 = 7$

S_2 = Sum of minors of diagonal elements

$= (6-2)+(4-1)+(6-2) = 4+3+4 = 11$

S_3 = Determinant $= 2(6-2) - 2(2-1) + 1(2-3)$

$= 8 - 2 - 1 = \underline{5}$

$\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0.$ $\quad \underline{\lambda = 1, 1, 5}$

Note. Set of all eigen values of A is called Spectrum of A

Properties

1) A and A^T have the same eigen values.

2) Eigen values of a diagonal, lower triangular, upper triangular matrices are the diagonal elements.

3) If λ is the eigen value of a matrix A then $1/\lambda$ is an eigen value of A^{-1}

4) If $\lambda_1, \lambda_2, \ldots \lambda_n$ are the eigen values of a matrix A then $\lambda_1^m, \lambda_2^m, \ldots \lambda_n^m$ are the eigen values of A^m.

5) If λ is an eigen value of A and k is any constant then $\lambda+k$ is an eigen value of $A+kI$

6) $k\lambda$ is an eigen value of kA.

Eigen vectors

The nontrivial solution of the equation $(A-\lambda I)x = 0$ is called eigen vector.

Probs

Find the eigen values and the corresponding eigen vectors of the matrix $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

Ans

$|A-\lambda I| = 0 \Rightarrow \lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$= \lambda^3 - 18\lambda^2 + 45\lambda - 0 = 0$

$\Rightarrow \lambda = 0 \quad \lambda = 3 \quad \lambda = 15$

$\lambda = 0$ Eigen vectors are obtained by $(A-\lambda I)x = 0$

$(A - 0I)x = 0$

$Ax = 0$

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$8x_1 - 6x_2 + 2x_3 = 0$

$-6x_1 + 7x_2 - 4x_3 = 0$

$2x_1 - 4x_2 + 3x_3 = 0$

$\frac{x_1}{24-14} = \frac{-x_2}{-32+12} = \frac{x_3}{56-36} = k$

$x_1 = 10k \quad x_2 = 20k \quad x_3 = 20k$

Eigen vector. $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

$\underline{\underline{\lambda = 3}}$. Eigen vector obtained by $(A - \lambda I)x = 0$ (9)

$$[A - 3I]x = 0.$$

$$\left\{\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}\right\} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$5x_1 - 6x_2 + 2x_3 = 0.$

$-6x_1 + 4x_2 - 4x_3 = 0$

$2x_1 - 4x_2 + 0x_3 = 0.$

$$\frac{x_1}{24-8} = \frac{-x_2}{-20+12} = \frac{x_3}{20-36} = k$$

$x_1 = 16k \qquad x_2 = +8k \qquad x_3 = -16k$

Eigen vector $\begin{bmatrix} 2 \\ +1 \\ -2 \end{bmatrix}$

$\lambda = 15$ Eigen vector obtained by $(A - 15I)x = 0$

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-7x_1 - 6x_2 + 2x_3 = 0$

$-6x_1 - 8x_2 - 4x_3 = 0$

$2x_1 - 4x_2 - 12x_3 = 0$

$$\frac{x_1}{24+16} = \frac{-x_2}{28+12} = \frac{x_3}{56-36} = k$$

$x_1 = 40k \qquad x_2 = -40k \qquad x_3 = 20k$

Eigen vector $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

2) Find the eigen values and corresponding eigen vectors of the given matrix $A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$

Ans: $|A - \lambda I| = 0 \Rightarrow \lambda^2 - S_1\lambda + S_2 = 0$

$S_1 \to$ Sum of diagonal elements

$S_2 \to$ Determinant

$\Rightarrow \lambda^2 + 7\lambda + 6 = 0$

$\Rightarrow \lambda = -6, -1$

Eigen vector corresponding to $\lambda = -6$.

$(A + 6I)X = 0 \Rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_1 + 2x_2 = 0.$ $\quad x_1 = -2x_2.$

$2x_1 + 4x_2 = 0$ $\quad$ Put $x_2 = 1$ $\quad x_1 = +2$

Eigen vector $\begin{bmatrix} +2 \\ -1 \end{bmatrix}$

Eigen vector corresponding to $\lambda = -1$

$[A - (-1)I]X = 0 \Rightarrow [A + I]X = 0.$

$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-4x_1 + 2x_2 = 0$ $\quad 2x_1 = x_2.$

$2x_1 - x_2 = 0.$ $\quad x_1 = 1 \quad x_2 = 2$

Eigen Vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

3) $A = \begin{bmatrix} 3 & -2 \\ 9 & -6 \end{bmatrix}$

Ans: $|A - \lambda I| = 0 \Rightarrow \lambda^2 - S_1\lambda + S_2 = 0$

$\lambda^2 - 3\lambda + (-18 + 18) = 0$

$\lambda^2 + 3\lambda = 0 \quad \lambda = 0, -3$ //

(10)

$\lambda = 0$ Eigen vector obtained by $(A - 0I)X = 0$

$$\begin{bmatrix} 3 & -2 \\ 9 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$3x_1 - 2x_2 = 0$ $\qquad 3x_1 = 2x_2$

$9x_1 - 6x_2 = 0$ $\qquad x_2 = \frac{3}{2}x_1$

Put $x_1 = 1$ $\quad x_2 = 3/2$

Eigen Vector $\begin{bmatrix} 1 \\ 3/2 \end{bmatrix}$

$\lambda = -3$ Eigen vector obtained by $(A - 3I)X = 0$

$$\begin{bmatrix} 6 & -2 \\ 9 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$6x_1 - 2x_2 = 0$ $\qquad 6x_1 = 2x_2 \quad x_2 = 3x_1$

$9x_1 - 3x_2 = 0$ $\qquad$ Put $x_1 = 1 \quad x_2 = 3$

Eigen Vector $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

---

4 $\begin{bmatrix} 4 & 2 & -2 \\ 2 & 5 & 0 \\ -2 & 0 & 3 \end{bmatrix}$

$\Rightarrow |A - \lambda I| = 0 \Rightarrow \lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$S_1 = 12$

$S_2 = 15 + (12-4) + (20-4) = 39$

$S_3 = 4(15) - 2(6) - 2(10) = 28$

$\lambda^3 - 12\lambda^2 + 39\lambda - 28 = 0$

$\lambda = 1, 4, 7$

Eigen value corresponding to $\lambda=1$ $(A-I)X=0$

$$\begin{bmatrix} 3 & 2 & -2 \\ 2 & 4 & 0 \\ -2 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$3x_1 + 2x_2 - 2x_3 = 0$

$2x_1 + 4x_2 + 0x_3 = 0$

$-2x_1 + 0x_2 + 2x_3 = 0$

$$\frac{x_1}{8} = \frac{-x_2}{4} = \frac{x_3}{12-4} = k$$

$x_1 = 8k$ $x_2 = -4k$ $x_3 = 8k$

eigen vector. $\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$

Eigen value corresponding to $\lambda = 4$

$$(A-4I)X=0 \Rightarrow \begin{bmatrix} 0 & 2 & -2 \\ 2 & 1 & 0 \\ -2 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$0x_1 + 2x_2 - 2x_3 = 0$

$2x_1 + x_2 + 0x_3 = 0$

$-2x_1 + 0x_2 - x_3 = 0$

$$\frac{x_1}{2} = -\frac{x_2}{4} = \frac{x_3}{-4} = k$$

$x_1 = 2k$ $x_2 = -4k$ $x_3 = -4k$

Eigen Vector $\begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$

Eigen Vector corresponding to $\lambda=7$

$$(A-7I)X=0. \Rightarrow \begin{bmatrix} -3 & 2 & -2 \\ 2 & -2 & 0 \\ -2 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-3x_1 + 2x_2 - 2x_3 = 0$

$2x_1 - 2x_2 + 0x_3 = 0$

$-2x_1 + 0x_2 - 4x_3 = 0$

$$\frac{x_1}{-4} = \frac{-x_2}{4} = \frac{x_3}{6-4} = k$$

$x_1 = -4k$ $x_2 = -4k$

$x_3 = 2k$

Eigen vector $\begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$

(11).

5 $\begin{bmatrix} 6 & 5 & 2 \\ 2 & 0 & -8 \\ 5 & 4 & 0 \end{bmatrix}$

$\Rightarrow$ $|A-\lambda I|=0 \Rightarrow \lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$S_1 = 6+0+0 = 6$

$S_2 = 32 + -10 + -10 = 12.$

$S_3 = 6(32) - 5(40) + 2(8) = 192 - 200 + 16 = 8$ //

$\lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0 \Rightarrow \lambda = 2, 2, 2.$

Eigen vector corresponding to $\lambda = 2.$

$(A - 2I)X = 0$

$\Rightarrow \begin{bmatrix} 4 & 5 & 2 \\ 2 & -2 & -8 \\ 5 & 4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$4x_1 + 5x_2 + 2x_3 = 0$

$2x_1 - 2x_2 - 8x_3 = 0$

$5x_1 + 4x_2 - 2x_3 = 0$

$\frac{x_1}{-40+4} = \frac{-x_2}{-32-4} = \frac{x_3}{-8-10} = k$

$x_1 = -36k \quad x_2 = 36k \quad x_3 = -18k$

Eigen Vector $x_1 = 2 \quad x_2 = -2 \quad x_3 = 1$

$\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

Eigen Space

The eigen vectors corresponding to one and the same eigen value λ of A together with '0' form a vectorspace called eigen space of A corresponding to that λ. Linearly independent eigen vectors form a basis for Eigen space.

Note

1) The sum of the elements of the diagonal of a matrix is called trace of the matrix.
The trace of a matrix A equals the sum of the eigen values of a matrix.

2) The determinant of a matrix A equals the product of the eigen values of A.

Algebraic Multiplicity and Geometric multiplicity

The order M_λ of an eigen value λ as a root of the characteristic polynomial is called Algebraic multiplicity.

The number m_λ of linearly independent eigen vectors corresponding to λ is called geometric multiplicity of λ. This m_λ is the dimension of the eigen space corresponding to this λ.

Pbm

1 Determine the algebraic multiplicity and geometric multiplicity of the following matrices.

1 $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$S_1 = 5$
$S_2 = 7$
$S_3 = 3$

$\Rightarrow$ $|A - \lambda I| = 0 \Rightarrow \lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0 \Rightarrow \lambda = 1, 1, 3$

Algebraic multiplicity $M_1 = 2$ & $M_3 = 1$

For $\lambda = 3$

$(A - 3I)X = 0 \Rightarrow \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-x_1 + x_2 + x_3 = 0$

$x_1 - x_2 + x_3 = 0$

$-2x_3 = 0.$

$\dfrac{x_1}{1+1} = \dfrac{-x_2}{-1-1} = \dfrac{x_3}{1-1} = k$

$x_1 = 2k \quad x_2 = 2k \quad x_3 = 0k.$

Eigen vector $\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Geometric multiplicity

Number of independent eigen vector corresponding to $\lambda = 3$ is $1 \Rightarrow M_3 = 1$

For $\lambda = 1$

$(A - I)X = 0. \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x_1 + x_2 + x_3 = 0$

$x_1 + x_2 + x_3 = 0$

put $x_2 = t_1 \quad x_3 = t_2$

$x_1 = -t_1 - t_2.$

put $t_1 = 1$, $t_2 = 0.$ eigen vector $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$x_1 = -1 \quad x_2 = 1 \quad x_3 = 0.$

put $t_1 = 0$, $t_2 = 1$ $\quad x_1 = -1$, $x_2 = 0$, $x_3 = 1$ eigen vector $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

These eigen vectors are linearly independent.

Geometric multiplicity of $\lambda = 1$ is $m_1 = 2$.

Diagonalization

If a square matrix A of order n×n has 'n' linearly independent eigen vectors. X is a model matrix which is formed by grouping the eigen vectors of A. Then A can be diagonalisable such that $X^{-1}AX = D$. And D is the Diagonal matrix with diagonal entries are eigen values.

$$X^{-1}AX = D$$
$$A = XDX^{-1}$$
$$A^2 = XD^2X^{-1}$$
$$\vdots$$
$$A^n = XD^nX^{-1}$$

Problems.

1 Find the matrix X which diagonalizes the matrix $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$. Verify that $X^{-1}AX = D$, a diagonal matrix

A is diagonalizable by the matrix X whose columns are linearly independent eigen vectors of A

$$A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$$

$|A - \lambda I| = 0 \Rightarrow \lambda^2 - s_1\lambda + s_2 = 0 \Rightarrow \lambda^2 - 7\lambda + 10 = 0 \Rightarrow \lambda = 2, 5$

When $\lambda = 2$

$$(A - 2I)x = 0 \qquad \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$2x_1 + x_2 = 0 \qquad x_2 = -2x_1 \qquad$ Put $x_1 = 1 \qquad x_2 = -2.$

Eigen vector for $\lambda = 2$ is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\underline{\lambda = 5}$ $(A-5I)x = 0$ $\begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-x_1 + x_2 = 0$ $x_1 = x_2$ put $x_1 = 1 \Rightarrow x_2 = 1$

eigen vector for $\lambda = 5$ is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Hence the matrix $X = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$

$X^{-1} = \begin{bmatrix} 1/3 & -1/3 \\ -2/3 & 1/3 \end{bmatrix}$

$X^{-1}AX = \begin{bmatrix} 1/3 & -1/3 \\ -2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} = D$

---

2. Diagonalise if possible $\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$

$\rightarrow$ $X = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

Quadratic form

The general quadratic form in two variable is $ax^2 + by^2 + 2hxy$

The corresponding matrix form is

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & h \\ h & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

ie $Q = X^T A X$.

The quadratic form in 3 variable is $ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx$. The corresponding matrix form is

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Transformation to Principal axes [Canonical form]

If you give a square matrix A first find the eigen vectors and find the normalised form of eigen vectors.

normalised $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{x_1}{\sqrt{x_1^2+x_2^2+x_3^2}} \\ \frac{x_2}{\sqrt{x_1^2+x_2^2+x_3^2}} \\ \frac{x_3}{\sqrt{x_1^2+x_2^2+x_3^2}} \end{bmatrix}$

then form the orthogonal matrix X by grouping the normalised eigen vectors, then diagonalise the matrix $X^{-1}AX = D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

The canonical form equal to $\lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2$

Pbms

1 Find out what type of conic section does following quadratic form represents and transform it to principal axes. $Q = 17x_1^2 - 30x_1x_2 + 17x_2^2 = 128$

→ $17x_1^2 - 30x_1x_2 + 17x_2^2 = 128$

$a = 17$ $b = 17$ $2h = -30$ $h = -15$

$A = \begin{bmatrix} a & h \\ h & b \end{bmatrix} = \begin{bmatrix} 17 & -15 \\ -15 & 17 \end{bmatrix}$

$|A - \lambda I| = 0 \Rightarrow \lambda^2 - 34\lambda - 64 = 0$ $\lambda = 2, 32$

Hence the quadratic form is $2y_1^2 + 32y_2^2 = 128$

$$\frac{y_1^2}{64} + \frac{y_2^2}{4} = 1 \Rightarrow \text{It is an ellipse.}$$

When $\lambda = 2$

$$(A - 2I)X = 0 \Rightarrow \begin{bmatrix} 15 & -15 \\ -15 & 15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$15x_1 - 15x_2 = 0 \Rightarrow x_1 = x_2$ put $x_1 = 1 \Rightarrow x_2 = 1$.

Eigen vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

normalizing we get $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

When $\lambda = 32$

$$(A - 32I)X = 0 \Rightarrow \begin{bmatrix} -15 & -15 \\ -15 & -15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-15x_1 - 15x_2 = 0 \Rightarrow x_2 = -x_1$. put $x_1 = 1 \Rightarrow x_2 = +1$.

Eigen vector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

normalizing $\begin{bmatrix} \frac{-1}{\sqrt{2}} \\ 1/\sqrt{2} \end{bmatrix}$

Model matrix

$$X = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Transforming into principal axes we have

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = XY = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$x_1 = \frac{1}{\sqrt{2}} y_1 - 1/\sqrt{2}\, y_2$ $\quad x_2 = \frac{1}{\sqrt{2}} y_1 + 1/\sqrt{2}\, y_2$.

2 Transform to canonical form and to principal axes

$4x_1^2 + 24x_1x_2 - 14x_2^2 = 20$.

$\rightarrow$ $x_1 = \frac{2}{\sqrt{5}}y_1 + \frac{1}{\sqrt{5}}y_2$ $\quad x_2 = \frac{1}{\sqrt{5}}y_1 - 2/\sqrt{5}\, y_2$.

3 $3.7x_1^2 + 3.2x_1x_2 + 1.3x_2^2 = 4.5$

$\rightarrow$ $x_1 = \frac{2}{\sqrt{5}}y_1 + 1/\sqrt{5}\, y_2$ $\quad x_2 = 1/\sqrt{5}\, y_1 - 2/\sqrt{5}\, y_2$.

4. Diagonalize the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$

$\rightarrow$ $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

5 Reduce the quadratic form to canonical form

$Q = x^2 + 3y^2 + 3z^2 - 2yz$

$\rightarrow$ $a = 1 \quad b = 3 \quad c = 3 \quad 2h = 0 \quad h = 0 \quad 2f = -2 \quad f = -1 \quad g = 0$

$$A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

$|A - \lambda I| = 0 \Rightarrow \lambda^3 - 7\lambda^2 + 14\lambda - 8 = 0 \qquad \lambda = 1, 2, 4.$

$\underline{\underline{\lambda = 1}}$ $\quad (A - I)x = 0$ $\quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\left.\begin{aligned} 0x + 0y + 0z &= 0 \\ 0x + 2y - z &= 0 \\ 0x - y + 2z &= 0 \end{aligned}\right\} \qquad \frac{x}{4-1} = \frac{-y}{0} = \frac{z}{0} = k.$$

$\Rightarrow x = 3k \quad y = 0 \quad z = 0.$

Eigen vector $\begin{bmatrix}1\\0\\0\end{bmatrix}$

normalised form $\begin{bmatrix}1\\0\\0\end{bmatrix}$.

When $\lambda = 2$

$(A-2I)X=0 \Rightarrow \begin{bmatrix}-1 & 0 & 0\\ 0 & 1 & -1\\ 0 & -1 & 1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

$-x+0y+0z=0$
$0x+y-z=0$
$0x-y+z=0$

$\frac{x}{0} = \frac{-y}{1} = \frac{z}{-1} = K$

$x=0 \quad y=-k \quad z=-k$

Eigen vector $\begin{bmatrix}0\\1\\1\end{bmatrix}$

normalised form $\begin{bmatrix}0\\ \frac{1}{\sqrt{2}}\\ 1/\sqrt{2}\end{bmatrix}$

$\lambda = 4$ $(A-4I)X=0 \Rightarrow \begin{bmatrix}-3 & 0 & 0\\ 0 & -1 & -1\\ 0 & -1 & -1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

$-3x+0y+0z=0$
$0x-y-z=0$
$0x-y-z=0$

$\frac{x}{0} = \frac{-y}{3} = \frac{z}{3} = k$

$x=0 \quad y=-3k \quad z=3k$.

Eigen vector $\begin{bmatrix}0\\-1\\1\end{bmatrix}$

normalized vector. $\begin{bmatrix}0\\-1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$

$X = \begin{bmatrix}1 & 0 & 0\\ 0 & 1/\sqrt{2} & -1/\sqrt{2}\\ 0 & 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}$

Transform to Principal axis $\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1/\sqrt{2} & -1/\sqrt{2}\\ 0 & 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}\begin{bmatrix}x_1\\y_1\\z_1\end{bmatrix}$

6 Diagonalize the matrix $A = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$ and find A^5

$\Rightarrow$ $|A-\lambda I| = 0 \Rightarrow \lambda^2 - 3\lambda + 2 = 0 \quad \lambda = 1, 2.$

$\underline{\underline{\lambda = 1}}$ $(A - I)X = 0 \Rightarrow \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$x_1 + 3x_2 = 0. \Rightarrow x_1 = -3x_2.$ $\quad x_2 = 1 \Rightarrow x_1 = -3$

Eigen vector $X_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

$\underline{\underline{\lambda = 2}}$ $(A - 2I)X = 0 \Rightarrow \begin{bmatrix} 0 & 3 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$

$3x_2 = 0 \quad x_2 = 0.$

x_1 can be taken any value $\quad x_1 = 1$

$X_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$

Model matrix $X = \begin{bmatrix} -3 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad X^{-1} = \begin{bmatrix} 0 & -1 \\ -1 & -3 \end{bmatrix}$

$X^{-1}AX = \begin{bmatrix} -3 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = D.$

$X^{-1}AX = D \Rightarrow A = XDX^{-1}$

$A^5 = XD^5X^{-1}$ $\quad D^5 = \begin{bmatrix} 1 & 0 \\ 0 & 32 \end{bmatrix}$

$A^5 = \begin{bmatrix} -3 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 32 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} -32 & -93 \\ 0 & -1 \end{bmatrix}$

→ The Symmetric matrix with all <u>eigen values</u> are <u>positive</u> is called a <u>positive definite</u> matrix

→ Positive <u>Semidefinite</u> matrix was defined as a Symmetric matrix with <u>non negative</u> <u>eigen values</u>

→ Negative Definite matrix is a Hermitian matrix (16)

all of whose eigen values are negative

→ A negative Semi definite matrix is a Hermitian matrix

all of whose eigen values are non positive

[Hermitian Square matrix is a complex square matrix that is equal to its own Conjugate transpose]

Orthogonal Transformations

A real matrix A is called Orthogonal if $A^T = A^{-1}$

Or $A \cdot A^T = I$

Orthogonal transformations are transformations $y = Ax$. Where A is an Orthogonal matrix.

→ Determinant of an Orthogonal matrix has the value +1 or -1

Proof $\det AB = \det A \cdot \det B$.

$\det A^T = \det A$.

$1 = \det(I) = \det(AA^{-1}) = \det(AA^T) = \det A \cdot \det A^T \cdot [\det(A)]^2$

$\therefore \det(A) = \pm 1$.

→ The eigen values of an Orthogonal matrix A are real or Complex conjugates in pairs and have absolute value 1.

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Eg: $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

- $A^T = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

$$A \cdot A^T = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & \cos\alpha\sin\alpha - \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha - \cos\alpha\sin\alpha & \sin^2\alpha + \cos^2\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\Rightarrow$ A Orthogonal

Multivariable Calculus - Differentiation

Partial derivatives

If f is a function of one variable then the derivative of f w.r.to x is denoted by $\frac{df}{dx}$.

If 'f' is a function of two variables x & y then the derivatives are called partial derivatives and partial derivative of f w.r.to x is denoted by $\frac{\partial f}{\partial x}$ or f_x.
partial derivative of f w.r.to y is denoted by $\frac{\partial f}{\partial y}$ or f_y.

Problems

1. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $z = x^4 \sin(xy^3)$

-> A

$$Z = x^4 \sin(xy^3).$$

$$\frac{\partial z}{\partial x} = x^4 \cos(xy^3) \cdot y^3 + \sin(xy^3) \times 4x^3$$

$$\frac{\partial z}{\partial y} = x^4 \cos(xy^3) \times 3xy^2.$$

2. $f(x,y) = 2x^3y^2 + 2y + 4x$ find $f_x(1,3)$ & $f_y(1,3)$

-> $f_x = 6x^2y^2 + 4$ $\quad f_x(1,3) = 58$

$f_y = 4x^3y + 2$ $\quad f_y(1,3) = 14$

3 $f(x,y,z) = x^3y^2z^4 + 2xy + z$ Compute f_x, f_y, f_z

→ $f_x = y^2z^4 + 2y$ $f_y = 2yx^3z^4 + 2x$ $f_z = 4x^3y^2z^3 + 1$

4 $f(\rho, \phi, \theta) = \rho^q \cos\phi \sin\theta$. find f_ρ, f_θ, f_ϕ

5 $z = e^{3x} \sin y$ find $\frac{\partial z}{\partial x}$ at $(x, 0)$ and $\frac{\partial z}{\partial y}(\log 3, 0)$

6 $f(x,y) = xe^{-y} + 5y$. find the slope of the surface $z = f(x,y)$ in the x-direction at $(2,5)$

→ Slope of z in the x direction $= \frac{\partial z}{\partial x}$

$= e^{-y}$

at $(2,5$ $= e^{-5}$

7 $f(x,y) = \sin(y^2 - 4x)$ find the rate of change of the surface $z = f(x,y)$ w.r.to y at the pt $(3,1)$ with x fixed

→ $\frac{\partial f}{\partial y} = \cos(y^2 - 4x) \times 2y$

at $(3,1)$ $= \cos(1 - 4 \times 3) \times 2$ $= 2\cos(-11)$ $= 2\cos 11$

8. $z = (x+y)^{-1}$ find $\frac{\partial z}{\partial x}$ at $(-1,4)$

$\frac{\partial z}{\partial x} = -\frac{1}{(x+y)^2}$ at $(-1,4)$ $= -1/9$

9. A pt moves along intersection of any thick paraboloid $z = x^2 + 3y^2$ and the plane $y = 1$ at what rate is z changing w.r.to x when the pt at $(3,1,12)$

→ Given $z = x^2 + 3y^2$ and $y = 1$ $\Rightarrow z = x^2 + 3$

$$\frac{\partial z}{\partial x} = 2x \quad \text{at } (3,1,12) = 2\times 3 = 6. \quad (2)$$

10 $f(x,y,z) = x^2y^4z^3 + xy + z^2 + 1$ find $f_x, f_y,$ and $f_z(1,2,3)$

→ $f_x(1,2,3) = 866$ $\quad f_y(1,2,3) = 865$ $\quad f_z(1,2,3) = 438$

11 If $f(x,y) = y^2e^x + y$ find f_{xyy}

→ $f_x = y^2e^x.$

$f_{xy} = 2ye^x$

$f_{xyy} = 2e^x.$

12. $f(x,y) = y^3e^{-5x}$ find f_{yyxx} at $(0,1)$

$f_y = 3y^2e^{-5x}.$

$f_{yy} = 6y\,e^{-5x}$

$f_{yyx} = -30y\,e^{-5x}$

$f_{yyxx} = 150y\,e^{-5x}.$

$f_{yyxx}(0,1) = 150$

Higher Order Partial derivatives

$$\frac{\partial^2 f}{\partial x^2} = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx}. \quad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy}$$

The last two partial derivatives are called mixed partial derivatives.

Differentiability

A function f of two variables x, y is said to be differentiable at (x_0, y_0) provided $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ both exists and

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\Delta f - f_x(x_0, y_0)\Delta x - f_y(x_0, y_0)\Delta y}{\sqrt{\Delta x^2 + \Delta y^2}} = 0$$

where $\Delta f = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$

A function f of three variables x, y, z is said to be differentiable at (x_0, y_0, z_0) if $f_x(x_0, y_0, z_0)$, $f_y(x_0, y_0, z_0)$, $f_z(x_0, y_0, z_0)$ exist and

$$\lim_{(\Delta x, \Delta y, \Delta z) \to (0,0,0)} \frac{\Delta f - f_x(x_0, y_0, z_0)\Delta x - f_y(x_0, y_0, z_0)\Delta y - f_z(x_0, y_0, z_0)\Delta z}{\sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}} = 0$$

where $\Delta f = f(x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z) - f(x_0, y_0, z_0)$

Problems

1 S.T $f(x,y) = x^2 + y^2$ is differentiable at $(0,0)$

→ $f_x = 2x$ $\quad f_y = 2y$

$f_x(0,0) = 0$ $\quad f_y(0,0) = 0$

$\Delta f = f(0 + \Delta x, 0 + \Delta y) - f(0,0) = \Delta x^2 + \Delta y^2$

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\Delta f - f_x(x_0, y_0)\Delta x - f_y(x_0, y_0)\Delta y}{\sqrt{\Delta x^2 + \Delta y^2}}$$

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\Delta x^2 + \Delta y^2}{\sqrt{\Delta x^2 + \Delta y^2}} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \sqrt{\Delta x^2 + \Delta y^2} = 0.$$

$\therefore$ f is diff at $(0,0)$

2) S.T $f(x,y,z) = x^2+y^2+z^2$ is differentiable at $(0,0,0)$

Theorem

1) If a function is differentiable at a point it is continuous at that point.

2) If all 1st order partial derivatives exist and are continuous at a point then f is differentiable at that point.

Pbms

3) S.T $f(x,y,z) = x+yz$ differentiable every where

→ $\frac{\partial f}{\partial x} = 1$ $\frac{\partial f}{\partial y} = z$ $\frac{\partial f}{\partial z} = y$ are defined and continuous every where. So f is diff everywhere.

4) S.T $f(x,y) = x^2 \sin y$ is differentiable everywhere.

5) S.T $f(x,y,z) = xy\sin z$ is differentiable every where.

Differentials

If $z = f(x,y)$ is differentiable at a point (x,y) then $\boxed{dz = f_x(x,y)dx + f_y(x,y)dy}$ is the total differential of z or f at (x,y).

If $w = f(x,y,z)$ then $\boxed{dw = f_x(x,y,z)dx + f_y(x,y,z)dy + f_z(x,y,z)dz}$ is called total differential of w at (x,y,z).

Change Δz in $z \simeq dz$

ie Change Δz in z is approximately the differential dz where dx is change in x and dy is change in y.

If $\Delta x, \Delta y$ are close to 0, the magnitude of the error in the approximation will be much smaller than the distance $\sqrt{\Delta x^2+\Delta y^2}$ b/w (x,y) and $(x+\Delta x, y+\Delta y)$

Problems

1) Find approximately the change in $z=xy^2$ at $(0.5, 1)$ to its value at $(0.503, 1.004)$. Compare the magnitude of the error in the approximation with the distance b/w $(0.5, 1)$, $(0.503, 1.004)$

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

$$= y^2 dx + 2xy\, dy$$

$dx = .003 \qquad dy = .004$

$\therefore dz = 1\times.003 + 2\times.5\times1\times.004 = .007$

Change Δz in z is $.007$

By actual calculation change Δz in z is

$.503(1.004)^2 - .5\times(1)^2 = .007032048$

Error $= .000032048$.

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Distance b/w pts $= \sqrt{(.003)^2 + (.004)^2} = .005$ (4)

$\therefore \dfrac{|dz - \Delta z|}{\sqrt{\Delta x^2 + \Delta y^2}} = \dfrac{.00032048}{.005} = .0064096 < 1/150.$

2 The lengths, Width and height of a Rectangular box are measured with an error atmost 5%. Find the maximum % error that result if these quantities are used to Calculate the diagonal of the box.

If x is length, y breadth, z height.

then $D = \sqrt{x^2+y^2+z^2}$

differential $dD = \dfrac{\partial D}{\partial x}dx + \dfrac{\partial D}{\partial y}dy + \dfrac{\partial D}{\partial z}dz$

$= \dfrac{1}{2\sqrt{x^2+y^2+z^2}}\,2x\,dx + \dfrac{1}{2\sqrt{x^2+y^2+z^2}}\,2y\,dy + \dfrac{1}{2\sqrt{x^2+y^2+z^2}}\,2z\,dz$

$$dD = \frac{x\,dx + y\,dy + z\,dz}{\sqrt{x^2+y^2+z^2}}$$

Given $\left|\dfrac{\Delta x}{x}\right| \le 0.05$ $\left|\dfrac{\Delta y}{y}\right| \le 0.05$ & $\left|\dfrac{\Delta z}{z}\right| \le 0.05$

$$\therefore \frac{\Delta D}{D} \approx \frac{dD}{D} = \frac{x\,dx + y\,dy + z\,dz}{\sqrt{x^2+y^2+z^2}\cdot\sqrt{x^2+y^2+z^2}}$$

$$= \frac{x\Delta x + y\Delta y + z\Delta z}{x^2+y^2+z^2}$$

$$= \frac{x^2\cdot\frac{\Delta x}{x} + y^2\cdot\frac{\Delta y}{y} + z^2\cdot\frac{\Delta z}{z}}{x^2+y^2+z^2}$$

$$\le \frac{0.05\,(x^2+y^2+z^2)}{x^2+y^2+z^2} = 0.05$$

$\therefore$ Max % of error in D is 5%.

Local linear approximation

f is a differentiable at a point (x_0, y_0)

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

is called local linear approximation to f at (x_0, y_0).

If f is function of three variables and f is differentiable at (x_0, y_0, z_0) then local linear approximation to f at (x_0, y_0, z_0) is

$$L(x,y,z) = f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0)$$

Pbms

1 Let $L(x,y)$ denote the local linear approximation to $f(x,y) = \sqrt{x^2+y^2}$ at $(3,4)$. Compare the error in approximating $f(3.04, 3.98)$ by $L(3.04, 3.98)$ with the distance b/w $(3,4)$, $(3.04, 3.98)$

$L(x,y) = f(3,4) + f_x(3,4)(x-3) + f_y(3,4)(y-4)$ 5

$= 5 + \frac{3}{5}(x-3) + 4/5(y-4)$

$L(3.04, 3.98) = 5 + 3/5 \times .04 + 4/5 \times -0.02 = 5.008$

$f(3.04, 3.98) = \sqrt{(3.04)^2 + (3.98)^2} \approx 5.00819.$

Error = .00019.

Distance b/w the pts $\approx \sqrt{(.04)^2 + (.02)^2} \approx .045$

Error less than 1/200 of the distance b/w the pts.

2. Find local linear approximation $f(x,y,z) = xyz$ at the pt P(1,2,3). Compare the error in approximating f by L at the specified pt Q(1.001, 2.002, 3.003) with the distance b/w P and Q.

$L(x,y,z) = 6 + 6(x-1) + 3(x-2) + 2(x-3)$

$L(1.001, 2.002, 3.003) = 6.018$

$f(1.001, 2.002, 3.003) = 6.018018006.$

Error = .000018

Distance = .00374165

Error < 1/200 of distance b/w the pts

3. Find local linear approximation L to function $f(x,y) = \frac{1}{\sqrt{x^2+y^2}}$ at (4,3). Compare the error in approximating f by L at the pt (3.92, 3.01) with distance b/w the pts.

Chain Rule

If $x = x(t)$ and $y = y(t)$ are differentiable at t and if $z = f(x,y)$ is differentiable at the pt $(x,y) = (x(t), y(t))$ then z is differentiable at t and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}.$$

If $x = x(t)$, $y = y(t)$, $z = z(t)$ are differentiable at t and $w = f(x,y,z)$ is differentiable at t

and $$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dt}$$

Problems

1. If $x = t^2$, $y = t^3$ where $z = x^2 y$ find $\frac{dz}{dt}$

→ $$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$= 2xy \cdot 2t + x^2 \cdot 3t^2$$

$$= 2 \times t^2 \times t^3 \times 2t + (t^2)^2 \times 3t^2 = 4t^6 + 3t^6 = 7t^6$$

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2. $w = \sqrt{x^2+y^2+z^2}$ $\quad x = \cos\theta \quad y = \sin\theta \quad z = \tan\theta$.

find $\frac{dw}{d\theta}$ when $\theta = \pi/4$ $\quad$ [Ans $\sqrt{2}$]

3. $z = \log(2x^2+y)$ $\quad x = \sqrt{t}$ $\quad y = t^{1/3}$ find dz/dt

Chain rule for Partial differentiation

If $x = x(u,v)$, $y = y(u,v)$ have 1st order partial derivatives at (u,v) and if z is differentiable at (x,y) then z has first order partial derivatives at (u,v) given by

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}.$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}.$$

Problems

1 Given $z = e^{xy}$ $x = 2u+v$ $y = \frac{u}{v}$ find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$.

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u} = ye^{xy} \cdot 2 + xe^{xy} \cdot \frac{1}{v}$$

$$= \frac{2u}{v} e^{(2u+v)\frac{u}{v}} + \frac{(2u+v)}{v} e^{(2u+v)\frac{u}{v}}.$$

$$= e^{(2u+v)\frac{u}{v}} \left[\frac{4u}{v} + 1 \right]$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v} = ye^{xy} \cdot 1 + xe^{xy} \cdot -\frac{u}{v^2}$$

$$= e^{xy} \left[y - \frac{ux}{v^2} \right]$$

$$= e^{(2u+v)\frac{u}{v}} \left[\frac{u}{v} - \frac{u(2u+v)}{v^2} \right].$$

$$= e^{(2u+v)\frac{u}{v}} \left[-\frac{2u^2}{v^2} \right]$$

2. $w = e^{xyz}$ $\quad x = 3u+v$ $\quad y = 3u-v$ $\quad z = u^2 v$

find $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$

$\rightarrow$

$$\frac{\partial w}{\partial u} = e^{xyz}\left[3yz + 3x\cancel{z} + 2xy\,uv\right]$$

$$\frac{\partial w}{\partial v} = e^{xyz}\left[yz - xz + xy \cdot u^2\right]$$

3. If $w = x^2+y^2-z^2$

$x = \rho \sin\phi \cos\theta$ $\quad y = \rho \sin\phi \sin\theta$ $\quad z = \rho\cos\phi$

find $\frac{\partial w}{\partial \rho}$ and $\frac{\partial w}{\partial \theta}$.

$\rightarrow$ $-2\rho\cos 2\phi,\ 0.$

4. $w = xy+yz$ $\quad y = \sin x$ $\quad z = e^x$.

find $\frac{dw}{dx}$

$\rightarrow$ $x.\sin x + e^x \sin x$.

5 $z = 3x^2y^3$ $\quad x = t^4$ $\quad y = t^3$ $\quad$ find $\frac{dz}{dt}$

6. $z = \sqrt{1+x-2xy^4}$ $\quad x = \log t,$ $\quad y = 2t$ $\quad$ find $\frac{dz}{dt}$

7. $z = 8x^2y - 2x + 3y$ $\quad x = uv$ $\quad y = u+v$ $\quad$ find $\frac{\partial z}{\partial u}$ & $\frac{\partial z}{\partial v}$

8 $w = 5\cos(xy) - \sin(xz)$ $\quad x = 1/t$ $\quad y = t$ $\quad z = t$ $\quad$ find $\frac{dw}{dt}$

9. find $\frac{\partial f}{\partial u}$ at $u=1, v=-2$ and $\frac{\partial f}{\partial v}$ at $u=1, v=-2$

Where $f = x^2y^2 - x + 2y$, $x = \sqrt{u}$, $y = uv^3$.

Theorem

If the equation $f(x,y)=c$ defines implicitly as differential function of x then

$$\frac{dy}{dx} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \quad \text{if } \frac{\partial f}{\partial y} \neq 0 .$$

Pbm

Given $x^3+y^2x-3=0$ find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

$\frac{\partial f}{\partial x} = 3x^2+y^2$

$\frac{\partial f}{\partial y} = 2xy$

$$\frac{dy}{dx} = -\frac{(3x^2+y^2)}{2xy}.$$

Theorem

If $f(x,y,z)=c$ define z implicitly as a differentiable function of x,y and if $\frac{\partial f}{\partial z} \neq 0$

$$\left.\frac{\partial z}{\partial x} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}} \;\&\; \frac{\partial z}{\partial y} = \frac{-\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial z}}\right\} \text{ if } \frac{\partial f}{\partial z} \neq 0 .$$

Pbm Given $x^2+y^2+z^2=1$ then find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$.

at $(2/3, 1/3, 2/3)$

$\frac{\partial z}{\partial x} = \quad = -x/z \qquad \frac{\partial z}{\partial y} = -y/z$

at the pt. $\frac{\partial z}{\partial x} = -1 \qquad \frac{\partial z}{\partial y} = -1$

Maxima and Minima of functions of two variables

(I) A function f of two variables is said to have a relative maximum at (x_0, y_0) if there is a disc covered at (x_0, y_0) such that $f(x_0, y_0) \geq f(x, y)$ for every points (x, y) in the disc and absolute max at (x_0, y_0) if $f(x_0, y_0) \geq f(x, y)$ for every points (x, y) in the domain of f.

(II) A function f of two variables is said to have a relative minimum at (x_0, y_0) if $f(x_0, y_0) \leq f(x, y)$ for every points (x, y) in the disc and Absolute minimum if $f(x_0, y_0) \leq f(x, y)$ for every points (x, y) in the domain of f.

Note: If f has a relative maximum or relative minimum at (x_0, y_0) then we say f has a relative extremum at a point (x_0, y_0).

Theorem.

If $f(x, y)$ has a relative extremum at a point (x_0, y_0) and if the 1st order partial derivative of f exist at this point and $f_x(x_0, y_0) = 0$, $f_y(x_0, y_0) = 0$ then the point (x_0, y_0) is called a critical point.

Let $p = f_x(x,y)$, $q = f_y(x,y)$
$r = f_{xx}(x,y)$ $s = f_{xy}(x,y)$ $t = f_{yy}(x,y)$

Thm

Let $D = rt - s^2$ then at a critical point (x_0, y_0), we say that f has

i) If $D > 0$ and $r > 0$, a relative minimum at (x_0, y_0).

ii) If $D > 0$ and $r < 0$ then f has a relative maximum at (x_0, y_0)

(iii). If $D < 0$ then f has a saddle point at (x_0, y_0) ie neither max or minimum.

(iv). If $D = 0$ then no conclusion can be made.

Problems

1 Find the relative extremum of $f(x,y) = 3x^2 - 2xy + y^2 - 8y$..

$p = f_x = 6x - 2y$ $q = f_y = -2x + 2y - 8$.

Critical points $f_x = 0$ & $f_y = 0$

$6x - 2y = 0$ & $-2x + 2y - 8 = 0. \Rightarrow x = 2, y = 6$

$r = f_{xx} = 6$ $t = f_{yy} = 2$ $s = f_{xy} = -2$

$rt - s^2$ at $(2,6) = 12 - 4 > 0$ $r = 6 > 0$.

f has a relative minimum at $(2,6)$

and minimum value is $f = 3(2)^2 - 2(2)(6) + 6^2 - 8 \times 6$
$= -24$ //.

2 Find the extremum of the function.

$f(x,y) = 4xy - x^4 - y^4$

$\rightarrow$ $f_x = 4y - 4x^3$. $f_y = 4x - 4y^3$

Critical point $f_x = 0$, $f_y = 0$.

$4y - 4x^3 = 0$ $4x - 4y^3 = 0$.

$y = x^3$ $4x - 4x^9 = 0$

$4x(1 - x^8) = 0$

$x = 0$ $x^8 = 1$

$x = 0 \Rightarrow y = 0$

$x = 1, -1$

$x = 1 \Rightarrow y = 1$

$x = -1 \Rightarrow y = -1$

r s t

$f_{xx} = -12x^2$ $f_{xy} = 4$ $f_{yy} = -12y^2$.

Points	r	t	s	$D = rt - s^2$.
$(0,0)$	0	0	4	$-16 \rightarrow$ Saddle point (0
$(1,1)$	-12	-12	4	128 Relative Maxima
$(-1,-1)$	-12	-12	4.	128

$(0,0) \rightarrow$ Saddle point

Relative Maximum at $(1,1)$ & $(-1,-1)$.

3) $f(x,y) = 2xy - x^3 - y^2$

$\rightarrow$ $(0,0) \rightarrow$ Saddle point, Relative maxima at $(2/3, 2/3)$

4) $f(x,y) = y^2 + xy + 4y + 2x + 3$.

5) $f(x,y) = x^2 + xy - 2y - 3x + 1$

6) $f(x,y) = x^2 + xy + y^2 - 6x$.

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Absolute extremum

Step 1: Find the critical points of f that lies in the interior of R

Step 2 Find all boundary points at which the absolute extreme can occur.

Step 3: Evaluate f(x,y) at these points.

Largest of these values is absolute maximum and smallest absolute minimum.

Pbm

1 Find the absolute maximum and minimum of $f(x,y) = 3xy - 6x - 3y + 7$ on a closed triangular region with vertices $(0,0)$, $(3,0)$ and $(0,5)$.

Step 1

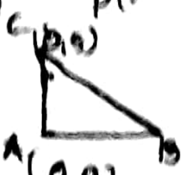
$f_x = 3y - 6$. $\quad f_x = 0 \Rightarrow y = 2$

$f_y = 3x - 3$. $\quad f_y = 0 \Rightarrow x = 1$.

Critical pt. $(1,2)$

Step 2

C (0,5)



A (0,0) B (3,0)

AB $y = 0$. $\quad f = -6x + 7$

$f_x = -6 \neq 0 \Rightarrow$ No critical pt.

AC $x = 0$. $\quad -3y + 7 \quad f_y = -3 \neq 0 \Rightarrow$ No critical pt

BC $\dfrac{y-0}{5-0} = \dfrac{x-3}{0-3}$ $\quad y = -\frac{5}{3}x + 5$

~~f(x,...)~~

$$f(x) = 3x\left(-\frac{5}{3}x+5\right) - 6x - 3[-5/3\,x+5] + 7$$

$$= -5x^2 + 15x - 6x + 5x - 15 + 7$$

$$= -5x^2 + 14x - 8$$

$f_x = 0 \implies -10x + 14 = 0 \quad x = 7/5$

$\implies y = -5/3 \times 7/5 + 5 = 8/3$

Critical pt $(7/5, 8/3)$

Step 3

(x,y)	$(1,2)$	$(7/5, 8/3)$	$(0,0)$	$(3,0)$	$(0,5)$
$f(x,y)$	1	$9/5$	7	-11	-8

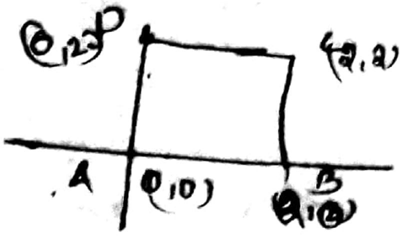
Absolute maxima at $(0,0)$, and absolute minema at $(3,0)$

2) $f(x,y) = x^2 - 3y^2 - 2x + 6y$ where R is the region bounded by the square with vertices. $(0,0)$, $(0,2)$, $(2,2)$, and $(2,0)$

→ $f_x = 2x - 2 \qquad f_x = 0 \implies x = 1$

$f_y = -6y + 6. \qquad f_y = 0 \implies y = 1.$

Critical point $(1,1)$.



AB $x=0$. $f_y = -6y+6$ $f_y = -6+\cdots$

$f(y) = -3y^2+6y$ $f_y = -6y+6$ $f_y=0 \Rightarrow y=1$.

Critical pt $(0,1)$.

BC $x=2$ $f(y) = 4-3y^2-4+6y$

$f_y = 0 \Rightarrow -6y+6 \Rightarrow y=1$

Critical point $(2,1)$

AB $y=0$ $f(x) = x^2-2x$

$f_x = 0 \Rightarrow 2x-2=0$ $x=1$

pt $(1,0)$.

DC $y=2$ $f(x) = x^2-12-2x+12$

$f_x = 0 \Rightarrow 2x-2=0$ $x=1$

pt $(1,2)$

(x,y)	$(0,0)$	$(2,0)$	$(2,2)$	$(0,2)$	$(1,1)$	$(0,1)$	$(2,1)$	$(1,0)$	$(1,2)$
$f(x,y)$	0	0	0	0	2.	3	3	-1	-1

Absolute Maximum at $(0,1)$ & $(2,1)$

Absolute Minimu at $(1,0)$ & $(1,2)$

Module IV

Multivariable Calculus - Integration

Double integrals

A double integral can be evaluated by two successive integrations. We evaluate it w.r.to one variable (treating the other variable as constant) and reduce it to an integral of one variable.

ie $\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx \right] dy$

$= \int_a^b \int_c^d f(x,y)\,dy\,dx$

[Rectangular region]

Problems

1 $\int_1^3 \int_2^4 40-2xy\,dy\,dx$

$= \int_1^3 \left[\int_2^4 40-2xy\,dy \right] dx$

$= \int_1^3 \left[40y - \frac{\cancel{2}xy^2}{\cancel{2}}\right]_2^4 dx = \int_1^3 80-12x\,dx$

$= \left. 80x - \frac{12x^2}{2} \right]_1^3 = 112$

2 Evaluate the double integral $\iint_R y^2x\,dA$ over the rectangle $R = \{(x,y) / -3 \le x \le 2,\ 0 \le y \le 1\}$

$\Rightarrow$ $= \iint_R y^2x\,dA = \int_{-3}^{2} \int_0^1 y^2x\,dy\,dx. = -5/6$

3 $\int_1^a \int_1^b \frac{1}{xy}\, dy\, dx \quad = \int_1^a \frac{1}{x}\, dx \int_1^b \frac{1}{y}\, dy$

$= (\log x)_1^a (\log x)_1^b = \log a \log b$

4 $\int_{\pi/2}^{\pi} \int_1^2 x \sin(xy)\, dy\, dx$

$\Rightarrow \int_{\pi/2}^{\pi} \left[\cancel{x} \frac{-\cos(xy)}{\cancel{x}} \right]_1^2 dx = \int_{\pi/2}^{\pi} -\cos 2x + \cos x\, dx$

$= -\frac{\sin 2x}{2} + \sin x \Big]_{\pi/2}^{\pi} = -1$

5 Evaluate. $\int_0^1 \int_{-x}^{x^2} y^2 x\, dy\, dx.$

$\rightarrow \int_0^1 x \frac{y^3}{3}\Big]_{-x}^{x^2} dx = \int_0^1 \frac{x^7}{3} + \frac{x^4}{3}\, dx = \frac{13}{120}.$

Fubini's theorem

Let R be a rectangle defined by the inequalities $a \le x \le b$, $c \le y \le d$. If $f(x,y)$ is continuous on this rectangle then,

$$\iint_R f(x,y)\, dA = \int_c^d \int_a^b f(x,y)\, dx\, dy = \int_a^b \int_c^d f(x,y)\, dy\, dx$$

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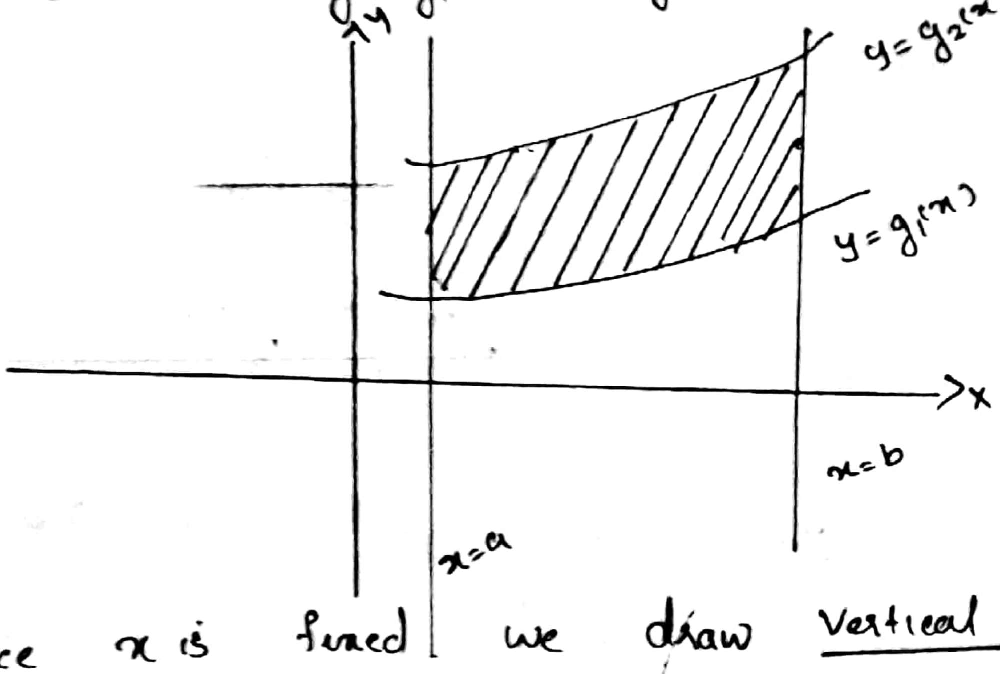
Double integral over non rectangular region

Type 1 Region : It is a region bounded on the left and right by the vertical line $x=a$ and $x=b$ and is bounded below and above by the curves $y=g_1(x)$ $y=g_2(x)$, $g_1(x) \le g_2(x)$ $a \le x \le b$

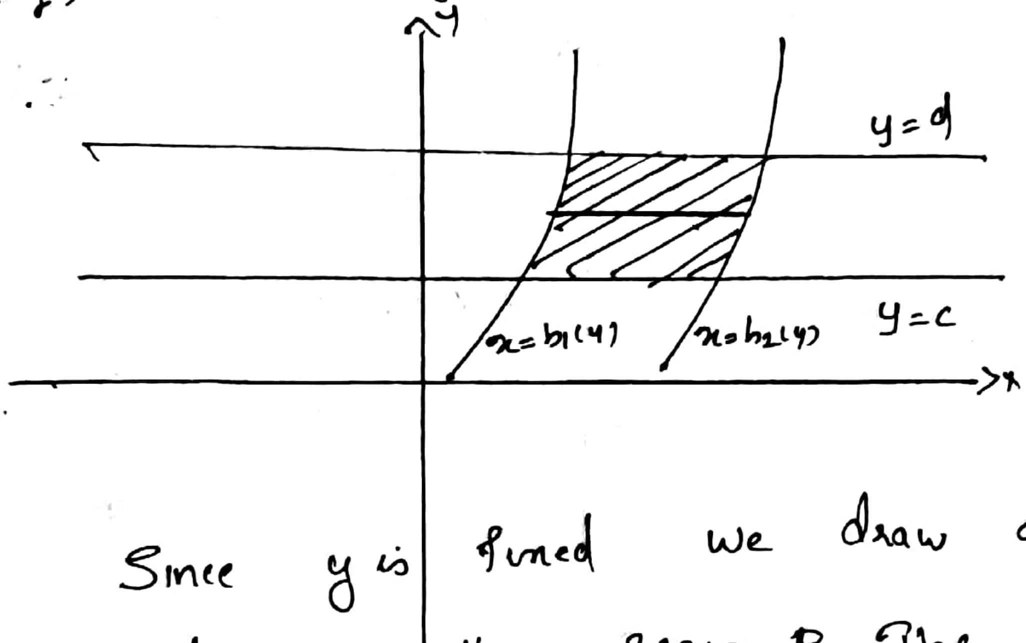


Since x is fixed we draw vertical line in the region R at an arbitrary fixed value. The line crosses the boundary of R twice. The lower point of the intersection is on the curve $y=g_1(x)$ higer point is on the curve $y=g_2(x)$. These two intersection determines lower and upper limit of y. Imagine move the line to left and then to right. left most position where the line intersect the region R is $x=a$ and the right position is $x=b$. This determines the limit of x.

x constant , y variable

Type 2 Region

It is a region bounded below and above by the horizontal lines $y=c$ and $y=d$ and bounded on left and right by the continuous curves $x=h_1(y)$ and $x=h_2(y)$ $\therefore$ $h_1(y) \le h_2(y)$ for $c \le y \le d$.



Since y is fixed we draw a horizontal line in the region R. The line also crosses the boundary twice. The left side is on the curve $x=h_1(y)$ and right side is on the curve $x=h_2(y)$. Move the line from bottom to top. ie from $y=c$ to $y=d$.

---

y — constant , x = variable

Problems

1 Evaluate $\iint_R xy\, dA$, R is enclosed b/w

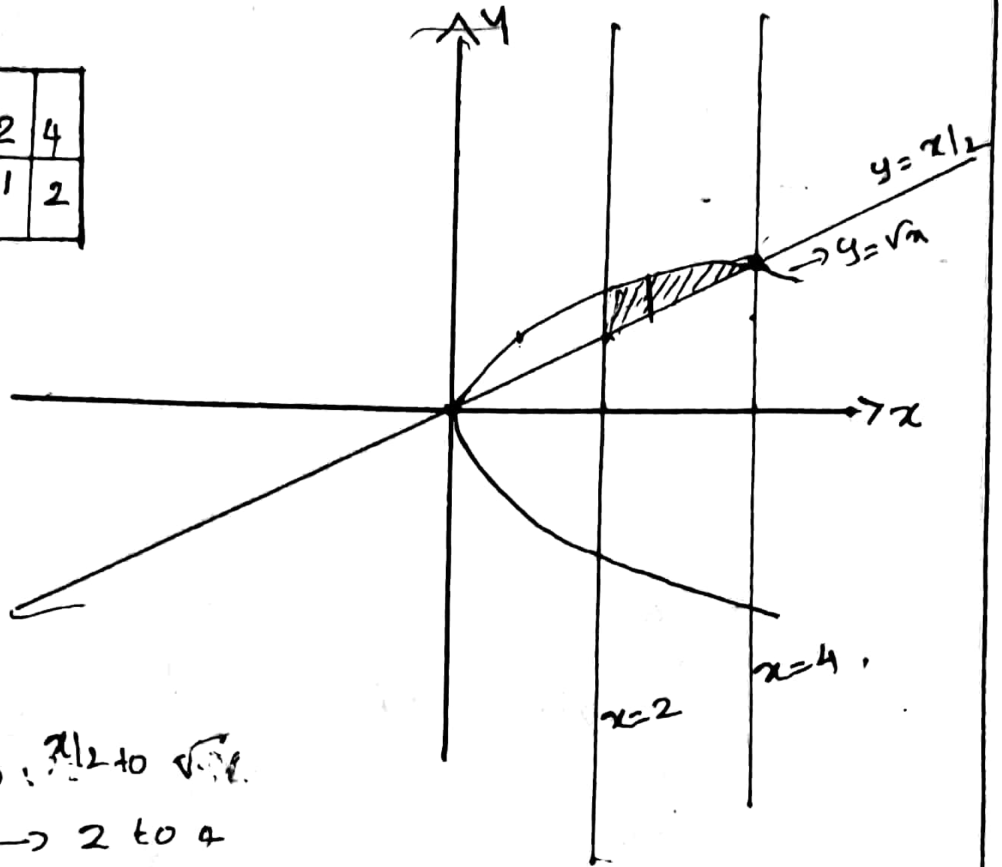
$y = x/2$, $y = \sqrt{x}$, $x = 2$, $x = 4$

→ $y = x/2$

x	0	2	4
y	0	1	2

$y = \sqrt{x}$

x	0	1	4
y	0	1	2



Type I

$y \to x/2$ to $\sqrt{x}$

$x \to 2$ to 4

$$\int_2^4 \int_{x/2}^{\sqrt{x}} xy\, dy\, dx = \int_2^4 x \left. \frac{y^2}{2} \right]_{x/2}^{\sqrt{x}} dx$$

$$= \int_2^4 \frac{x^2}{2} - \frac{x^3}{8}\, dx = \frac{11}{6}.$$

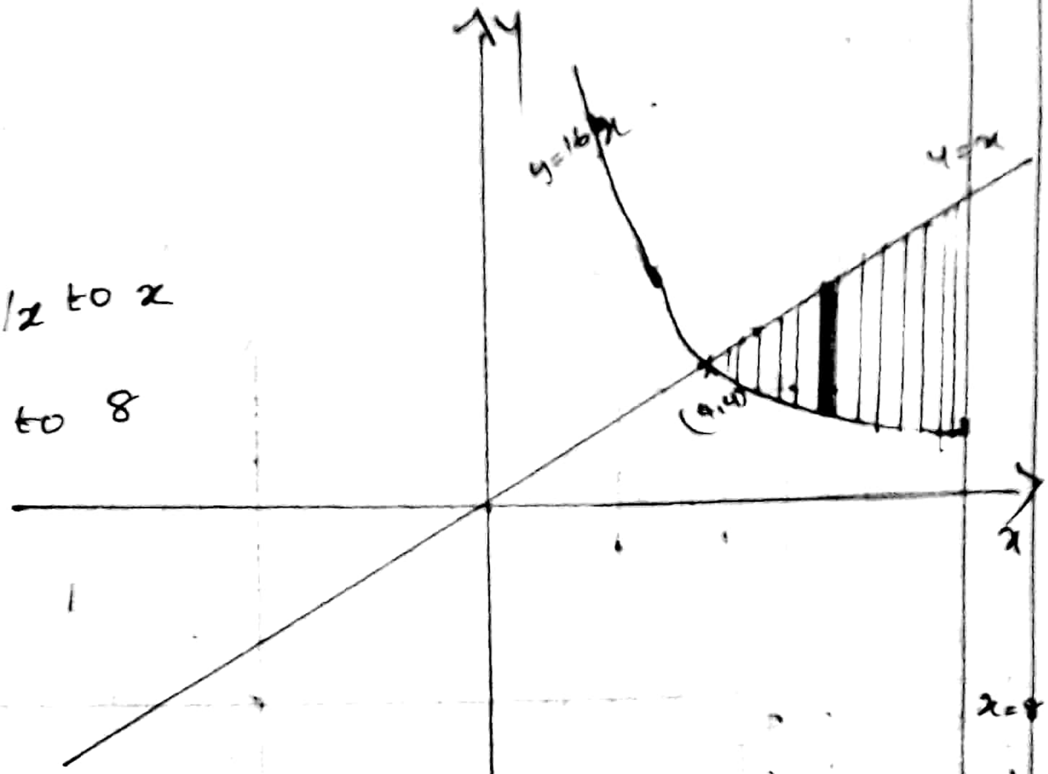
2 Evaluate $\iint_R x^2\, dA$. bounded by $y = 16/x$, $y = x$, $x = 8$.

⇒ $y = \frac{16}{x}$

2	2	4	8
4	8	4	2

$y = x$

x	0	4	8
y	0	4	8



$y \to 16/x$ to x

$x \to 4$ to 8

$$\int_4^8 \int_{16/x}^{x} x^2 \, dy \, dx = \int_4^8 x^2 y \Big]_{16/x}^{x} dx$$

$$= \int_4^8 x^3 - 16x \, dx = 576.$$

---

3 Evaluate $\iint_R 2x - y^2$ over the rectangular region is enclosed b/w $y = -x+1$, $y = x+1$, $y = 3$

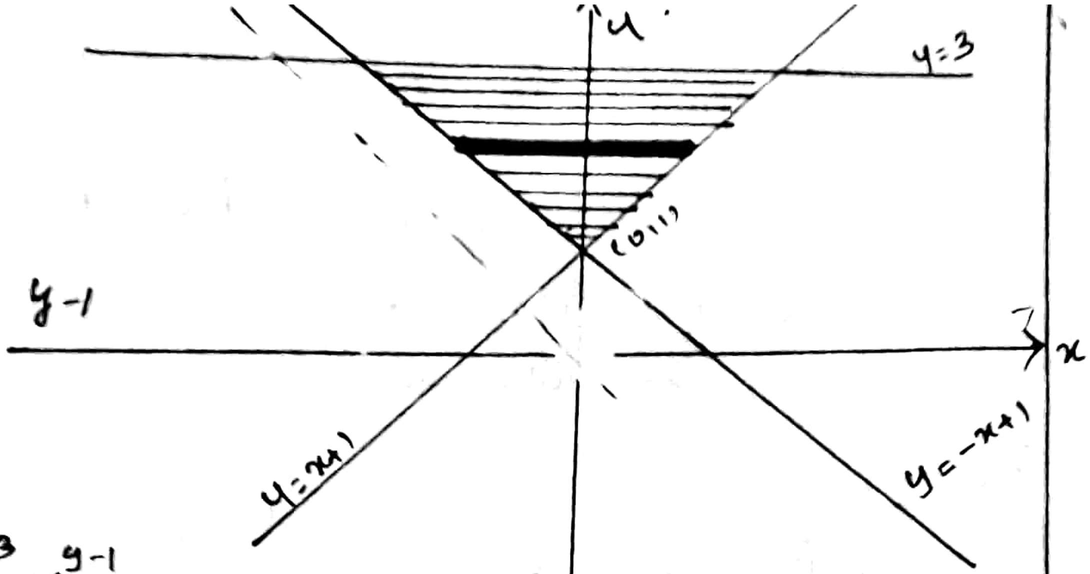
$\Rightarrow$ $y = -x+1$

x	0	1	2
y	+1	0	-1

$y = x+1.$

x	0	1	2
y	1	2	3

4



Type 2

$x \to 1-y$ to $y-1$

$y \to 1$ to 3

$$\int_1^3 \int_{1-y}^{y-1} (2x - y^2)\, dx\, dy = \int_1^3 \left[x^2 - y^2 x \right]_{1-y}^{y-1} dy$$

$$= \int_1^3 2y^2 - 2y^3 dy = \left[\frac{2y^3}{3} - \frac{2y^4}{4} \right] = -\frac{68}{3}$$

4 Use a double integral find the area of Region R enclosed b/w a parabola $y = \frac{x^2}{2}$ and the line $y = 2x$

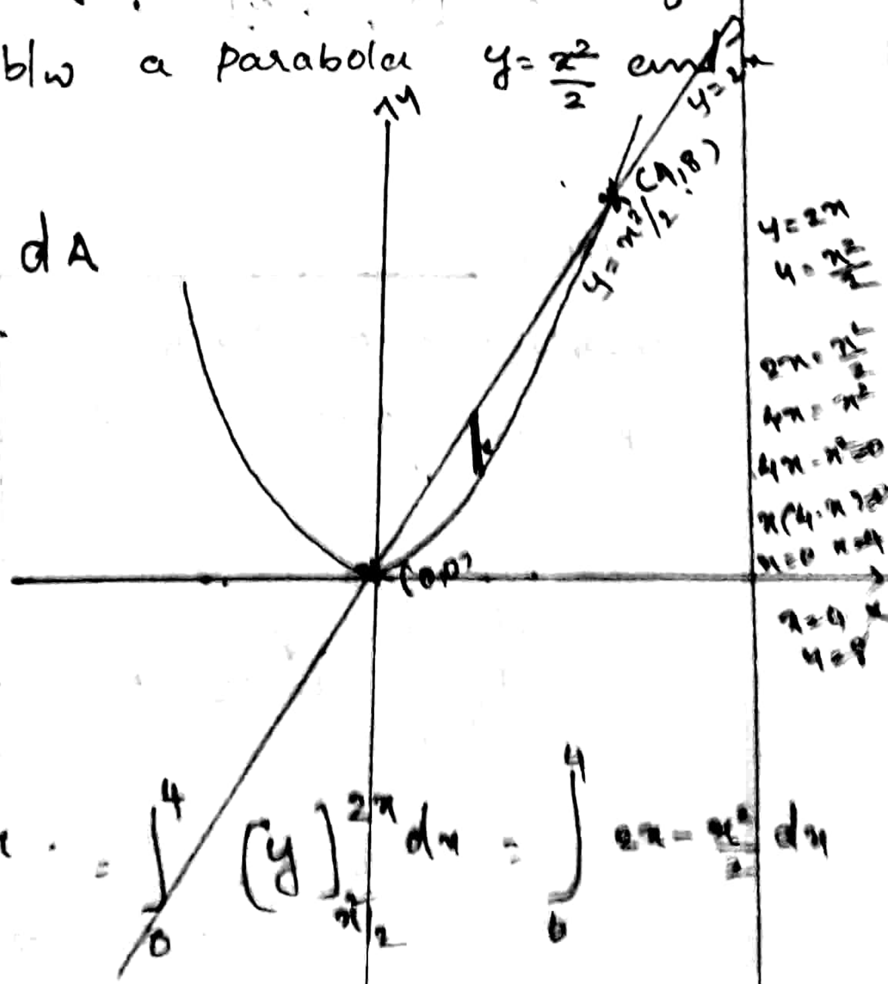
$$\text{AREA} = \iint_R dA$$

$y = \frac{x^2}{2}$

x	0	2	6
y	0	2	3

$y = 2x$

x	0	1	2
y	0	2	4



$y = 2x$
$y = \frac{x^2}{2}$
$2x = \frac{x^2}{2}$
$4x = x^2$
$4x - x^2 = 0$
$x(4-x) = 0$
$x=0 \quad x=4$
$x=4$
$y=8$

$y \to \frac{x^2}{2}$ to $2x$

$x \to 0$ to 4.

$$\text{Area} = \int_0^4 \int_{x^2/2}^{2x} dy\, dx = \int_0^4 \left[y \right]_{x^2/2}^{2x} dx = \int_0^4 2x - \frac{x^2}{2}\, dx$$

$$= \frac{16}{3}$$

Reversing the order of integration

Sometimes the evaluation of a double integral can be simplified by reversing the order of integration.

Problems

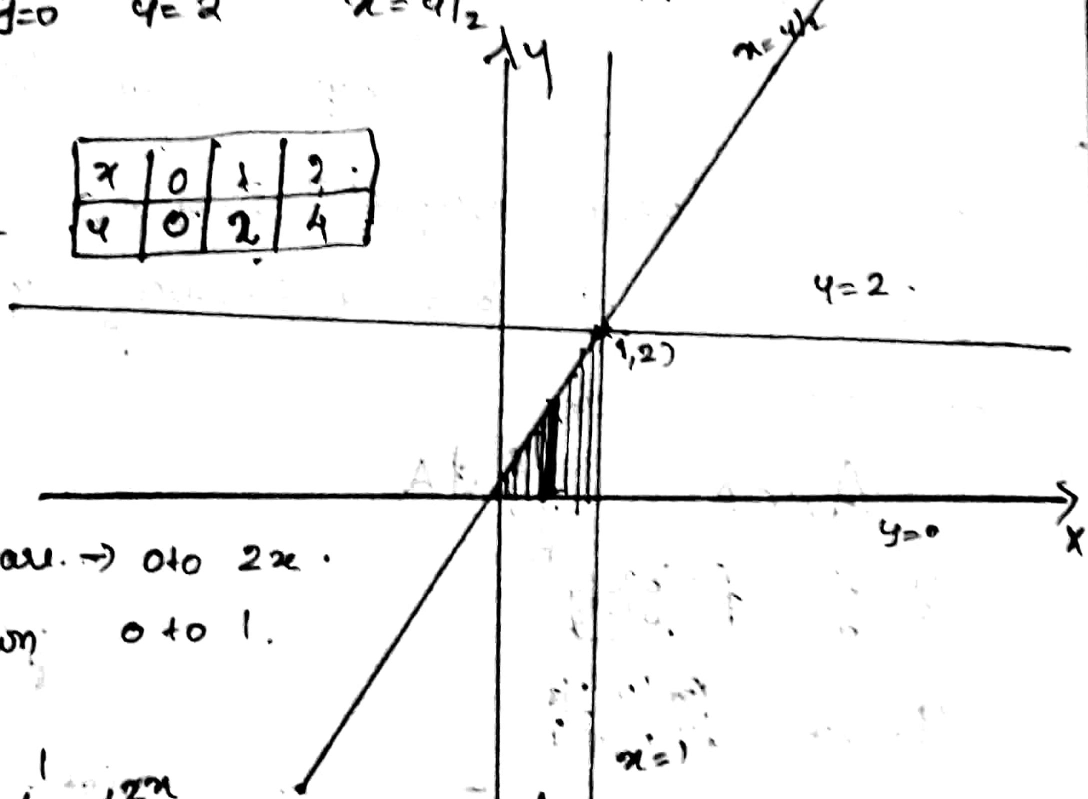
1 Evaluate integral $\int_0^2 \int_{y/2}^{1} e^{x^2}\, dx\, dy$

Type 2 region changes to Type 1 region

$y=0$ $y=2$ $x=y/2$ $x=1.$

$x = y/2$

x	0	1	2
y	0	2	4



Type 1

$y \to$ var. $\to$ 0 to $2x$.

$x \to$ con 0 to 1.

$$\int_0^1 \int_0^{2x} e^{x^2}\, dy\, dx = \int_0^1 e^{x^2} (y)_0^{2x}\, dx$$

$$= \int_0^1 e^{x^2}\, 2x\, dx = \left(e^{x^2}\right)_0^1 = \underline{\underline{e-1}}$$

5

2. Sketch the region of integration and evaluate the integral $\int_1^2 \int_y^{y^2} dx\, dy$ by changing the order of integration.

→ Type 2 Region <u>changes to</u>, Type 1 Region

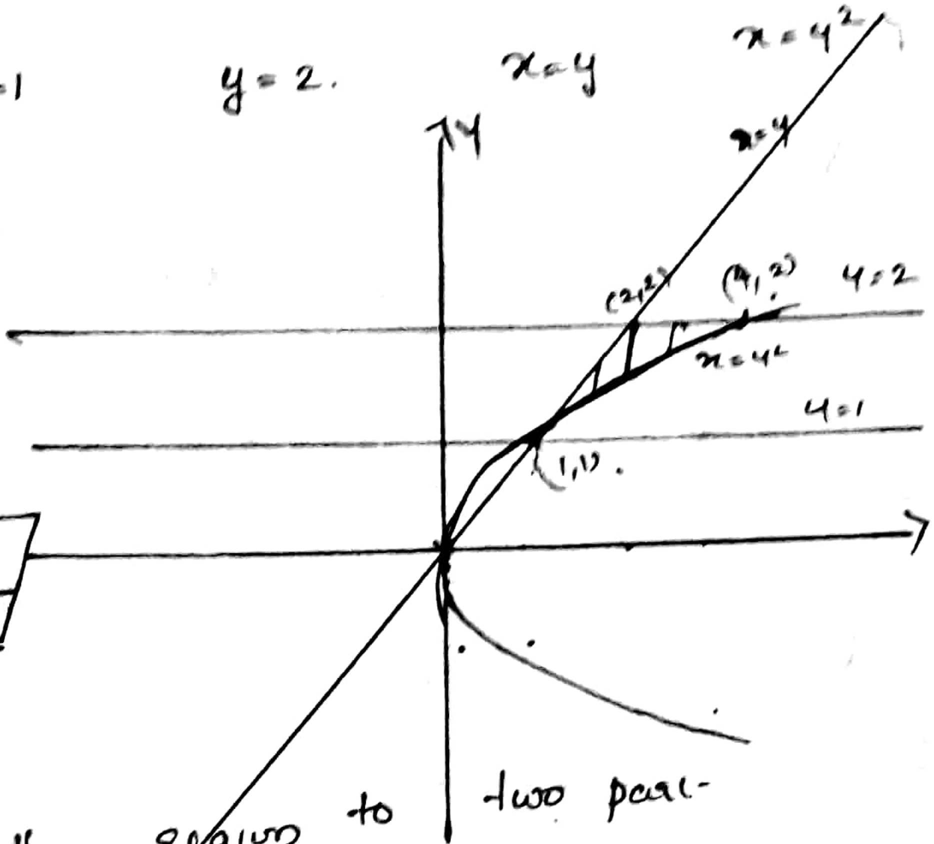
$y=1 \qquad y=2. \qquad x=y \qquad x=y^2$

$x=y$

x	0	1
y	0	1

$x=y^2$

x	0	1	1
y	0	1	-1



Split the region to two part

1st part. $y \to \sqrt{x}$ to x

$x \to 1$ to 2.

2nd part $y \to \sqrt{x}$ to 2

$x \to 2$ to 4.

$$\iint_R dx\,dy = \int_1^2 \int_{\sqrt{x}}^{x} dy\,dx + \int_2^4 \int_{\sqrt{x}}^{2} dy\,dx.$$

$$= \int_1^2 (y)_{\sqrt{x}}^{x}\,dx + \int_2^4 (y)_{\sqrt{x}}^{2}\,dx.$$

$$= \int_1^2 x-\sqrt{x} + \int_2^4 2-\sqrt{x}\,dx = \left[\frac{x^2}{2} - \frac{x^{3/2}}{3/2}\right]_1^2 + \left[2x-\frac{x^{3/2}}{3/2}\right]_2^4$$

$$= 5/6.$$

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Volume $= \iint f(x,y)\, dA$ Where $z = f(x,y)$

1) Find the volume of solid bounded by the cylinder $x^2+y^2=4$ $y+z=4$ $z=0$.

$\Rightarrow$ $y+z=4$ $z=4-y$

Volume $= \iint f(x,y)\, dA$

$= \iint 4-y \, dA.$

$x^2+y^2=4$

$y \to -\sqrt{4-x^2}$ to $\sqrt{4-x^2}$ $x \to -2$ to 2

$$V = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 4-y \, dy \, dx.$$

$$= \int_{-2}^{2} \left[4y - \frac{y^2}{2}\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx.$$

$$= \int_{-2}^{2} 4\sqrt{4-x^2} - \left(\frac{4-x^2}{2}\right) - \left[-4\sqrt{4-x^2} - \left(\frac{4-x^2}{2}\right)\right] dx$$

$$= \int_{-2}^{2} 8\sqrt{4-x^2}\, dx = 8\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}(x/2)\right]_{-2}^{2}$$

$$= 8\left[\sqrt{4-4} + 2\sin^{-1}(1) - \left(-\sqrt{4-4} + 2\sin^{-1}(-1)\right)\right]$$

$$= 8\left[2\sin^{-1}(1) + 2\times\sin^{-1}(1)\right] = 32\sin^{-1}(1)$$

$$= 32\times\pi/2$$

$$= 16\pi$$

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Triple Integrals

A Single integral of a function f(x) is defined over a finite closed interval on the x-axis, and a double integral of a function f(x,y) is defined over a finite closed region R in the xy-plane.

A triple integral of f(x,y,z) over a closed solid region G in the xyz-co-ordinate system.

$$\boxed{\text{Volume} = \iiint dv}$$

Pbms

1 Evaluate $\iiint_G 12xy^2z^3\,dv$ over the rectangular blocks G defined by the inequalities

$-1 \le x \le 2,\ 0 \le y \le 3,\ 0 \le z \le 2.$

$$\int_{-1}^{2}\int_{0}^{3}\int_{0}^{2} 12\,xy^2z^3\,dz\,dy\,dx$$

$$= \int_{-1}^{2}\int_{0}^{3} 12xy^2 \frac{z^4}{4}\Big]_0^2\,dy\,dx$$

$$\int_{-1}^{2}\int_{0}^{3} 48\,xy^2\,dy\,dx = \int_{-1}^{2} 48\,x\frac{y^3}{3}\Big]_0^3\,dx$$

$$= 432\int_{-1}^{2} x\,dx$$

$$= 432\left[\frac{x^2}{2}\right]_{-1}^{2} = \underline{648}$$

2. $\int_{0}^{1}\int_{-1}^{y^2}\int_{-1}^{z} yz\,dx\,dz\,dy$

$$= \int_{0}^{1}\int_{-1}^{y^2} [xyz]_{-1}^{z}\,dz\,dy$$

$$= \int_{0}^{1}\int_{-1}^{y^2} (yz^2 + yz)\,dz\,dy$$

$$\int_0^1 y\frac{z^3}{3} + y\frac{z^2}{2}\Big]_{-1}^{y^2} dy$$

$$= \int_0^1 \frac{y^7}{3} + \frac{y^5}{2} + y/3 - y/2 \, dy$$

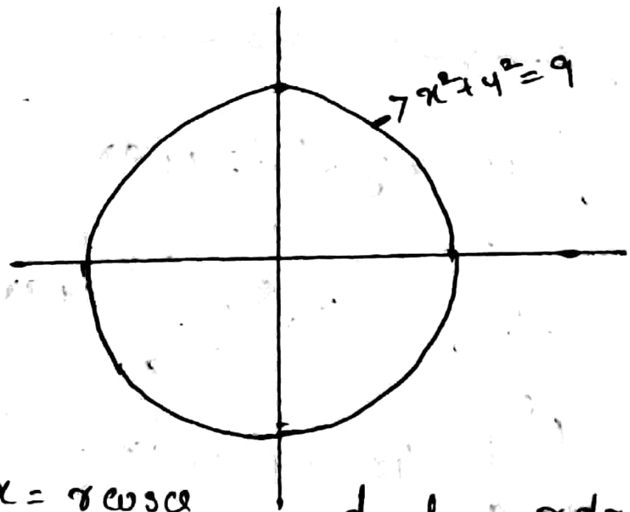
$$= \frac{y^8}{8\times 3} + \frac{y^6}{6\times 2} + \frac{y^2}{2\times 3} - \frac{y^2}{2\times 3}\Big]_0^1 = 1/24.$$

---

3. Use a triple integral to find the volume of the solid within the cylinder $x^2+y^2=9$ and between the planes $z=1$ and $x+z=5$.

→ Volume $= \iiint dV$

Here $z \longrightarrow 1$ to $5-x = 5-r\cos\theta$.



Sub $x = r\cos\theta$ $\quad dx\,dy = r\,dr\,d\theta$

$y = r\sin\theta$.

$r \rightarrow 0$ to 3

$\theta \rightarrow 0$ to 2π

$$\therefore V = \int_0^{2\pi}\int_0^3\int_1^{5-r\cos\theta} dz \cdot r\,dr\,d\theta.$$

$$\int_0^{2\pi}\int_0^{3}[z]_1^{5-r\cos\theta} \cdot r\,dr\,d\theta$$

$$\int_0^{2\pi}\int_0^{3}[5-r\cos\theta-1]\ r.\ dr\,d\theta$$

$$\int_0^{2\pi}\int_0^{3}[4-r\cos\theta]r\ dr\,d\theta$$

$$\int_0^{2\pi}\int_0^{3} 4r - r^2\cos\theta\ dr\,d\theta.$$

$$= \int_0^{2\pi}\left[\frac{4r^2}{2}-\frac{r^3}{3}\cos\theta\right]_0^3 d\theta$$

$$= \int_0^{2\pi} 18-9\cos\theta\ d\theta$$

$$= \left.18\theta-9\sin\theta\right]_0^{2\pi} = \underline{36\pi}.$$

4. Evaluate $\iiint xyz\ dv$ when G is the solid in the first octant that is bounded by the parabolic cylinder $z=3-x^2$ and the planes $z=0$ $y=x$ and $y=0$

→ <u>Limits</u>

$z \to 0$ to $3-x^2$.

$y \to 0$ to x

$x \to 0$ to $\sqrt{3}$

[$y=0,\ y=x \Rightarrow x=0$
$z=0\ \ z=3-x^2 \Rightarrow x^2=3$
$x=\sqrt{3}$]

$$\iiint_G xyz\ dv = \int_0^{\sqrt{3}}\int_0^{x}\int_0^{3-x^2} xyz\ dz\,dy\,dx$$

$$\int_0^{\sqrt{3}} \int_0^{x} xy \left.\frac{z^2}{2}\right]_0^{3-x^2} dy\, dx$$

$$\frac{1}{2}\int_0^{\sqrt{3}} \int_0^{x} xy\,(3-x^2)^2\, dy\, dx$$

$$\frac{1}{2}\int_0^{\sqrt{3}} \int_0^{x} xy\,(9-6x^2+x^4]\, dy\, dx.$$

$$\frac{1}{2}\int_0^{\sqrt{3}} \int_0^{x} 9xy - 6x^3y + x^5y\; dy\, dx$$

$$\int_0^{\sqrt{3}} \left[9x\frac{y^2}{2} - 6x^3\frac{y^2}{2} + x^5\frac{y^2}{2}\right]_0^{x} dx$$

$$\frac{1}{4}\int_0^{\sqrt{3}} 9x^3 - 6x^5 + x^7\, dx = \frac{27}{32}.$$

---

5 Use a triple integral to find the volume of the solid in the first Octant bounded by the Co-ordinate planes and the plane $3x+6y+4z=12$

$\rightarrow$ $V = \iiint dv.$

$$Z = \frac{12-3x-6y}{4}$$

$Z=0 \Rightarrow \frac{12-3x-6y}{4} = 0 \Rightarrow y = \frac{4-x}{2}.$

$Z=0,\ y=0 \Rightarrow \frac{4-x}{2} = 0 \Rightarrow x = 4.$

Limits

$z \rightarrow 0$ to $\frac{12-3x-6y}{4}$

$y \rightarrow 0$ to $\frac{4-x}{2}$

$x \rightarrow 0$ to 4

$$V = \int_0^4 \int_0^{\frac{4-x}{2}} \int_0^{\frac{12-3x-6y}{4}} dz\, dy\, dx$$

$$= \int_0^4 \int_0^{\frac{4-x}{2}} \frac{12-3x-6y}{4}\, dy\, dx.$$

$$\frac{1}{4} \int_0^4 \int_0^{\frac{4-x}{2}} 12-3x-6y\, dy\, dx$$

$$= \frac{1}{4} \int_0^4 \left. 12y - 3xy - \frac{6y^2}{2} \right|_0^{\frac{4-x}{2}} dx$$

$$= \frac{1}{4} \int_0^4 12\frac{(4-x)}{2} - 3x\frac{(4-x)}{2} - \frac{6\left(\frac{4-x}{2}\right)^2}{2}.$$

$$= \frac{1}{4} \int_0^4 24 - 6x - 6x + \frac{3x^2}{2} - 3\left(\frac{4-2x+\frac{x^2}{4}}{}\right) dx$$

$$\frac{1}{4} \int_0^4 12 - 6x + \frac{3x^2}{4}\, dx$$

$$\frac{1}{4} \left[12x - \frac{6x^2}{2} + \frac{3x^3}{3 \times 4} \right]_0^4 = 4$$

Mass of Lamina

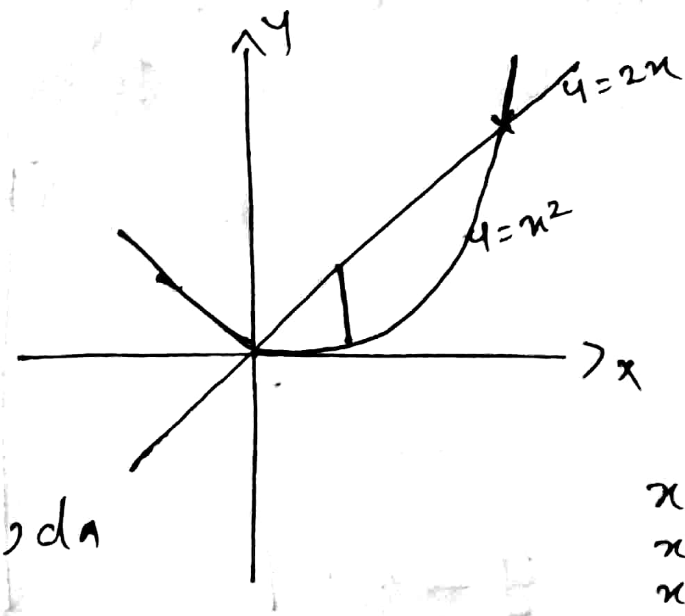
If $\rho(x,y)$ is a continuous density function of a lamina in the plane region R, then mass of lamina is

$$m = \iint_R \rho(x,y)\, dA$$

Pbms

1) Find the mass of the region that is bounded by the line $y = 2x$ and the parabola $y = x^2$ if the density function is $\rho(x,y) = x$.

→



$y \to x^2$ to $2x$.

$x \to 0$ to 2

$y = x^2$
$y = 2x$
$x^2 = 2x$
$x^2 - 2x = 0$
$x(x-2) = 0$
$x = 0,\ x = 2$

$$M = \int_0^2 \int_{x^2}^{2x} \rho(x,y)\, dA$$

$$= \int_0^2 \int_{x^2}^{2x} x \, dy\, dx. \quad = \int_0^2 [xy]_{x^2}^{2x} dx$$

$$= \int_0^2 2x^2 - x^3 dx \quad = \left.\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^2 = \underline{\underline{4/3}}$$

9

Centre of mass of Lamina

centre of mass $(\bar{x}, \bar{y})$

$$\bar{x} = \frac{M_y}{M}, \quad \bar{y} = \frac{M_x}{M}$$

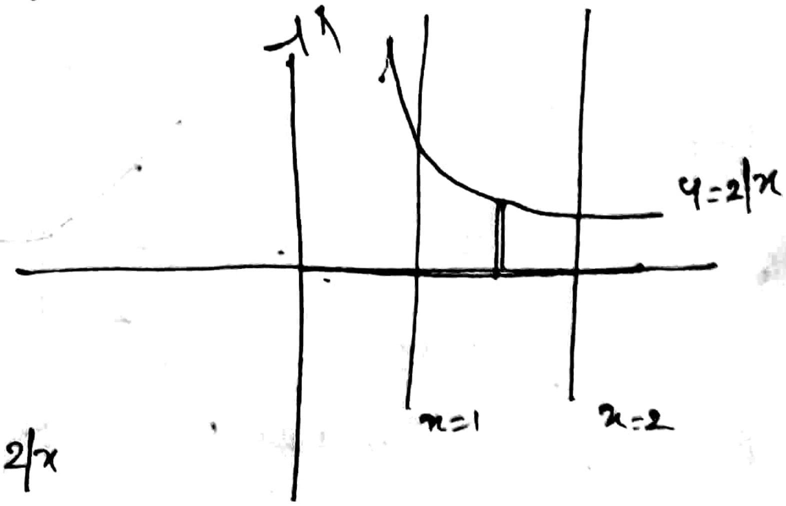
$$M_x = \iint_R y\, \rho(x,y)\, dA$$

$$M_y = \iint_R x\, \rho(x,y)\, dA.$$

Pbm

1) Find the mass and center of mass of the lamina bounded by $y = 2/x$, $y=0$, $x=1$, $x=2$ with density $\rho = kx^2$.

→



$y \to 0$ to $2/x$

$x \to 1$ to 2

$$\text{Mass } M = \int_1^2 \int_0^{2/x} \rho(x,y)\, dA$$

$$= \int_1^2 \int_0^{2/x} kx^2\, dy\, dx$$

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$$\int_1^2 \int_0^{2/x} kx^2 \, dy \, dx$$

$$\int_1^2 kx^2 y \Big]_0^{2/x} dx$$

$$\int_1^2 2k\,x \, dx = 2k \frac{x^2}{2}\Big]_1^2$$

$$= \underline{\underline{3k}}.$$

$$M_x = \iint_R y\, \rho(x,y)\, dA.$$

$$= \int_1^2 \int_0^{2/x} y \cdot kx^2 \, dy \, dx$$

$$= \int_1^2 kx^2 \cdot \frac{y^2}{2}\Big]_0^{2/x} dx$$

$$= \int_1^2 2k \, dx$$

$$= 2k\, [x]_1^2 = 2k.$$

$$M_y = \iint_R x\, \rho(x,y)\, dA$$

$$= \int_1^2 \int_0^{2/x} x\, kx^2 \, dy \, dx = \int_1^2 kx^3 [y]_0^{2/x} dx$$

$$= \int_1^2 2kx^2 \, dx$$

$$= 2k\left(\frac{x^3}{3}\right]_1^2$$

$$= 2k(7/3) = \underline{\underline{\frac{14}{3}k}}$$

$$\bar{x} = \frac{M_y}{M} = \frac{\frac{14}{3}k}{3k} = \frac{14}{9}$$

$$\bar{y} = \frac{M_x}{M} = \frac{2k}{3k} = 2/3$$

Centre of mass $= \left(\frac{14}{9}, 2/3\right)$

Double integrals

Area $= \iint dA$

$\boxed{dA = dx\,dy}$

Volume $= \iint z\,dA$

Mass of Lamina $M = \iint \rho(x,y)\,dA$

$\rho \rightarrow$ density

Centre of mass $= (\bar{x}, \bar{y})$

$$\bar{x} = \frac{M_y}{M} \qquad \bar{y} = \frac{M_x}{M}$$

$$M_y = \iint x\,\rho(x,y)\,dA$$

$$M_x = \iint y\,\rho(x,y)\,dA$$

Triple integrals

Volume $= \iint dv$

$\boxed{dv = dx\,dy\,dz}$

Polar co-ordinates.

Circle $x^2+y^2=r^2$

Put $x=r\cos\theta$, $y=r\sin\theta$, $dA=r\,dr\,d\theta$.

$r \to 0$ to r. (Eg: $x^2+y^2=4$
$r \to 0$ to 2)

$\theta \to 0$ to 2π.

Type 1 Region

$x \to$ constant, $y \to$ Variable

Type 2 - Region

$x \to$ Variable, y constant

(1)

Deepamol P.S.

I

Infinite Series

(1)

(6) Definition :

An infinite series is an expression that can be written in the form $\sum_{k=1}^{\infty} u_k = u_1 + u_2 + u_3 + \cdots + u_k + \cdots$

The numbers $u_1, u_2, u_3 \ldots$ are called the terms of the series.

Eg:- consider the decimal $0.3333\ldots$

This can be viewed as the infinite series

$0.3 + 0.03 + 0.003 + \ldots$ or $\frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots$

A sequence of partial sums of a series $\sum_{n=1}^{\infty} a_n$ is defined as the sequence $\{S_n\}$ where

$S_n = a_1 + a_2 + \cdots + a_n, \quad n = 1, 2, 3 \ldots$

eg:- consider the series $0.3 + 0.03 + 0.003 + \ldots$

then $S_1 = 0.3$

$S_2 = 0.3 + 0.03$

$S_3 = 0.3 + 0.03 + 0.003$

$\ldots$

$S_n = (0.3 + 0.03 + 0.003 + \cdots + 0.000\ldots03)$

$= \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots + \frac{3}{10^n}$

(18)

Convergence of infinite series

(2) Let $\{S_n\}$ be the sequence of partial sums of the series $u_1+u_2+u_3+\cdots+u_k+\cdots$. If the sequence $\{S_n\}$ converges to a limit S, then the series is said to converge to S, and S is called the sum of the series. It is denoted by $S=\sum_{k=1}^{\infty} u_k$

If the sequence of partial sums diverges, then the series is said to diverge.

A divergent series has no sum.

Eg: Determine whether the series $1-1+1-1+\cdots$, converges or diverges. If it converges, find the sum.

Here $S_1 = 1$

$S_2 = 1-1=0$

$S_3 = 1-1+1=1$

$S_4 = 1-1+1-1=0$

Thus the sequence of partial sum is $1,0,1,0,1,0\ldots$

This is a divergent sequence.

Hence the given series is also divergent and

Consequently has no sum

Geometric Series

Infinite series of the ~~sum~~ form $\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + \dots + ar^k + \dots$ $(a \neq 0)$ is called geometric series. The number 'r' is called the ratio for the series

Eg: * $1 + 2 + 4 + 8 + \dots + 2^k + \dots$ Here $a = 1$ & $r = 2$

* $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} + \dots + (-1)^{k+1} \frac{1}{2^k} + \dots$

Here $a = 1/2$ $r = -1/2$

Thereorem

A geometric series $\sum_{k=0}^{\infty} ar^k = a + ar + \dots + ar^k + \dots$ $(a \neq 0)$ converges if $|r| < 1$ and diverges if $|r| \geq 1$.

If the series converges, then the sum is $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$

* Determine whether the series converges, and if so find its sum

(1) $\sum_{k=0}^{\infty} \frac{5}{4^k}$

$\sum_{k=0}^{\infty} \frac{5}{4^k} = 5 + \frac{5}{4} + \frac{5}{4^2} + \dots + \frac{5}{4^k} + \dots$ Geometric series

Here $a = 5$ & $r = \frac{1}{4}$

Since $|r| = \left|\frac{1}{4}\right| < 1$, the given G.S is conv

and Sum is $\frac{a}{1-r} = \frac{5}{1-1/4} = \frac{20}{3}$

(2) Find the rational number represented by repeat decimal $0.784784784\ldots$

$\rightarrow 0.784784784\ldots = 0.784 + 0.000784 + 0.000000784 \ldots$

$= \frac{784}{10^3} + \frac{784}{10^6} + \frac{784}{10^9} + \ldots$

This is a geometric Series with $a = \frac{784}{10^3}$, $r = \frac{1}{10^3}$

Here $|r| < 1$ ∴ The g. Series Converges.

Sum $= \frac{a}{1-r} = \frac{0.784}{1-0.001} = \frac{.784}{.999} = \frac{784}{999}$

* Find all values of x for which the Series

$$3 - \frac{3x}{2} + \frac{3x^2}{4} - \frac{3x^3}{8} + \ldots + \frac{3(-1)^k}{2^k} x^k \ldots$$

converges and find the Sum of the Series for those values of x.

$\rightarrow$ This is a geometric series with $a = 3$, $r = -$

it converges if $\left|\frac{-x}{2}\right|<1$, or equivalently when $|x|<2$

when the series converges its sum is,

$$\sum_{K=0}^{\infty} 3\left(\frac{-x}{2}\right)^{K} = \frac{3}{1-\left(-\frac{x}{2}\right)} = \frac{6}{2+x}$$

Harmonic Series

An infinite series of the form $\sum_{K=1}^{\infty} \frac{1}{K}$.

$1+\frac{1}{2}+\frac{1}{3}+\cdots\cdots$ is called Harmonic Series. This Series is divergent

· Convergence Tests

I Comparison Test

Theorem :- Let $\sum_{K=1}^{\infty} a_K$ and $\sum_{K=1}^{\infty} b_K$ be series with nonnegative terms and Suppose that

$$a_1 \le b_1,\ a_2 \le b_2,\ a_3 \le b_3,\ \ldots,\ a_K \le b_K, \ldots$$

a) if Σb_K converges, then Σa_K also converges

b) if Σa_K diverges, then Σb_K also diverges

p-Series

An infinite Series $\sum_{K=1}^{\infty} \frac{1}{K^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$ converges if $p > 1$ and diverges if $0 < p \leq 1$

Problem

* Use the comparison test to determine whether the following Series converges or diverge

1) $\sum_{K=1}^{\infty} \frac{1}{\sqrt{K} - \frac{1}{2}}$

→ consider the Series $\sum_{K=1}^{\infty} \frac{1}{\sqrt{K}}$ which is diverges (It is a p-Series $p = 1/2 < 1$) Also $\frac{1}{\sqrt{K} - \frac{1}{2}} > \frac{1}{\sqrt{K}}$ for $K = 1, 2, \ldots \infty$

Hence by comparison test, the given Series is divergent.

2) $\sum_{K=1}^{\infty} \frac{1}{2K^2 + K}$

→ We have $\sum_{K=1}^{\infty} \frac{1}{2K^2} = \frac{1}{2} \sum_{K=1}^{\infty} \frac{1}{K^2}$ and $\sum_{K=1}^{\infty} \frac{1}{K^2}$ is a convergent p-Series $(p = 2)$

Also $\frac{1}{2k^2+k} < \frac{1}{2k^2}$ for $k = 1, 2 \ldots$

Hence by comparison test the given series is convergent. (7)

Limit comparison Test

Let Σa_k and Σb_k be series with positive terms and suppose that $\rho = \lim_{k\to\infty} \frac{a_k}{b_k}$

If ρ is finite and $\rho > 0$, then the series both converge or both diverge

* use limit comparison test determine whether the series is convergent or divergent

1) $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}+1}$

Let $a_k = \frac{1}{\sqrt{k}+1}$ & $b_k = \frac{1}{\sqrt{k}}$ (Σb_k is a divergent series)

$$\rho = \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\sqrt{k}}{\sqrt{k}+1} = \lim_{k\to\infty} \frac{1}{1+\frac{1}{\sqrt{k}}}$$

$$= 1$$

ρ is finite and positive. Therefore by limit comparison test the given series diverges

(24)

2) $\sum_{k=1}^{\infty} \frac{1}{2k^2+k} = \sum_{k=1}^{\infty} \frac{1}{2k^2(1+\frac{1}{2k})}$

Let $\sum b_k = \sum \frac{1}{2k^2}$ which is Convergent

$\rho = \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{1}{1+1/2k} = 1$ finite & positive $\therefore$ by limit Comparison test, the given series is Convergent.

Limit comparison test

Let $\sum a_k$ and $\sum b_k$ be Series with positive terms and suppose that $\rho = \lim_{k\to\infty} \frac{a_k}{b_k}$

If ρ is finite and $\rho>0$, then the series both converge or both diverge

* use limit comparison test determine whether the Series is convergent or divergent

1) $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}+1}$

$\frac{1}{\sqrt{k}+1} = \frac{1}{\sqrt{k}(1+\frac{1}{\sqrt{k}})}$

$k^{1/2}$, $1/2<1$ PSe

Let $a_k = \frac{1}{\sqrt{k}+1}$ & $b_k = \frac{1}{\sqrt{k}}$ ($\sum b_k$ is divergent s

$\rho = \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\sqrt{k}}{\sqrt{k}+1} = \lim_{k\to\infty} \frac{1}{1+}$

ρ is finite and +ve. Therefore by limit comp

Test the given series diverges.

(2) $\sum_{k=1}^{\infty} \frac{1}{2k^2+k} = \sum_{k=1}^{\infty} \frac{1}{2k^2\left[1+\frac{1}{2k}\right]}$ (9)

Let $\sum b_k = \sum \frac{1}{2k^2}$ which is convergent

$$\rho = \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{1}{1+1/2k} = 1 \text{, finite \& }$$

positive. ∴ By limit comparison test, the given series is convergent.

(3) $\sum_{k=1}^{\infty} \frac{3k^3-2k^2+4}{k^7-k^3+2}$

$$= \sum_{k=1}^{\infty} \frac{3k^3\left[1-\frac{2}{3k}+\frac{4}{3k^3}\right]}{k^7\left[1-\frac{1}{k^4}+\frac{2}{k^7}\right]}$$

Take $b_k = \frac{3k^3}{k^7} = \frac{3}{k^4}$

$\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{3}{k^4}$ converges (P series)

$$\rho = \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{1-2/3k+\frac{4}{3k^2}}{1-\frac{1}{k^4}+\frac{2}{k^7}}$$

$= 1$ finite & Non zero

$\therefore$ By limit comparison test, the given series is convergent

Q.

Test the convergence of the series $\sum_{k=1}^{\infty} \frac{1}{3^k+11}$

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$$\frac{1}{3^k+11} < \frac{1}{3^k}$$

$\frac{1}{3^k}$ is a geometric series $a = \frac{1}{3}$ & $r = \frac{1}{3} < 1$

$\therefore \sum_{k=1}^{\infty} \frac{1}{3^k}$ is convergent.

Hence by comparison test $\sum_{k=1}^{\infty} \frac{1}{3^k+11}$ is also convergent.

* $\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{8k^2-3k}}$.

$$a_k = \frac{1}{\sqrt[3]{8k^2-3k}} = \frac{1}{(8k^2-3k)^{1/3}}$$

$$= \frac{1}{(8k^2)^{1/3}\left[1-\frac{3k}{8k^2}\right]^{1/3}}$$

$$= \frac{1}{8^{1/3}k^{2/3}\left(1-\frac{3}{8k}\right)}$$

$$= \frac{1}{2k^{2/3}\left(1-\frac{3}{8k}\right)}.$$

Take $b_k = \frac{1}{2k^{2/3}}$ [$\sum b_k$ is a p series with $p<1$ Hence divergent

$$\rho = \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{1}{\left(1-\frac{3}{8k}\right)^{1/3}} = 1$$ finite &

Positive. Hence $\sum a_k$ is also divergent. by limit comparison test.

(11)

* $\sum_{k=1}^{\infty} \frac{1}{(2k+3)^{17}}$

$$a_k = \frac{1}{(2k+3)^{17}} = \frac{1}{k^{17}\left(2+\frac{3}{k}\right)^{17}}$$

Take $b_k = \frac{1}{k^{17}}$ $\Rightarrow \sum b_k$ converges

$$\rho = \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{1}{(2+3/k)^{17}}$$

$$= \frac{1}{2^{17}}$$, finite and +ve

$\therefore$ By limit comparison test the given series $\sum a_k$ is also convergent.

L'Hospital's Rule

If $\lim_{x\to a} \frac{f(x)}{g(x)}$ takes the indeterminate forms $\left(\frac{0}{0}, \frac{\infty}{\infty}\right)$

and $f(x)$ & $g(x)$ have derivatives of all order then,
$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$ provided the limit exists.

Again if $\frac{f'(x)}{g'(x)}$ takes indeterminate forms, then

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = \lim_{x\to a} \frac{f''(x)}{g''(x)}$$ provided the limit

exist finitely.

Eg:- $\lim_{x\to 2} \frac{x^2-4}{x-2}$

$$\Rightarrow \lim_{x\to 2} \frac{\frac{d}{dx}(x^2-4)}{\frac{d}{dx}(x-2)} = \lim_{x\to 2} \frac{2x}{1} = \underline{\underline{4}}$$

Aliter

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x+2)(x-2)}{(x-2)} = 2+2 = \underline{\underline{4}}$$

<u>Note</u>

* comparison test only applies to Series with non-negative terms.

<u>Ratio Test:-</u>

Let $\sum u_k$ be a Series with positive terms and suppose that, $\rho = \lim_{k\to\infty} \frac{u_{k+1}}{u_k}$.

[Try this test when u_k involves factorials or k^{th} powers]

(i) if $\rho < 1$, the Series converges

(ii) if $\rho > 1$ or $\rho = \infty$ the Series diverges

(iii) if $\rho = 1$, the Series may converge or diverge so that another test must be tried.

(1) Test whether the series converge or diverge

$$\sum_{K=1}^{\infty} \frac{1}{K!}$$

(15)

$$\rho = \lim_{K\to\infty} \frac{u_{K+1}}{u_K} = \lim_{K\to\infty} \frac{\frac{1}{(K+1)!}}{1/K!} = \lim_{K\to\infty} \frac{K!}{(K+1)!}$$

$$= \lim_{K\to\infty} \frac{K!}{(K+1)K!}$$

$$= \lim_{K\to\infty} \frac{1}{K+1} = 0 < 1$$

Hence the given series is convergent by Ratio Test.

(ii) $$\sum_{K=1}^{\infty} \frac{K}{2^K}$$

$$\rho = \lim_{K\to\infty} \frac{\frac{K+1}{2^{K+1}}}{K/2^K} = \lim_{K\to\infty} \frac{K+1}{K} \frac{2^K}{2^{K+1}} = \frac{1}{2} \lim_{K\to\infty} \frac{K+1}{K}$$

$$= \frac{1}{2} \lim_{K\to\infty} \frac{K(1+1/K)}{K}$$

$$= 1/2 < 1$$

Given series is convergent.

(iii) $$\sum_{K=1}^{\infty} \frac{K^K}{K!}$$

$$\rho = \lim_{K\to\infty} \frac{(K+1)^{K+1}}{(K+1)!} \cdot \frac{K!}{K^K}$$

$$= \lim_{K\to\infty} \frac{(K+1)^{K+1}}{K^K} \frac{K!}{(K+1)K!}$$

$$= \lim_{k\to\infty} \frac{(k+1)^k (k+1)}{k^k (k+1)}$$

$$= \lim_{k\to\infty} \left(\frac{k+1}{k}\right)^k = \lim_{k\to\infty} (1+1/k)^k$$

$$= e > 1$$

$\therefore$ the Given Series is divergent.

(iv) $\sum_{k=3}^{\infty} \frac{(2k)!}{4^k}$

$$\lim_{k\to\infty} \frac{(2(k+1))!}{4^{k+1}} \cdot \frac{4^k}{(2k)!}$$

$$\lim_{k\to\infty} \frac{(2k+2)!}{(2k)!} \cdot \frac{1}{4}$$

$$= \lim_{k\to\infty} \frac{(2k+2)(2k+1)(2k)!}{(2k)!}$$

$= \underline{\underline{\infty}}$ $\therefore$ the Series diverges

(v) $\sum_{k=1}^{\infty} \frac{1}{2k-1}$

$$S = \lim_{k\to\infty} \frac{1}{2(k+1)-1} \cdot 2k-1$$

$$= \lim_{k\to\infty} \frac{1}{2k+1} \cdot 2k-1 = \lim_{k\to\infty} \frac{2k(1-\frac{1}{2k})}{2k(1+\frac{1}{2k})}$$

$= \underline{1}$ Test fail

* $\sum_{k=1}^{\infty} \frac{1}{1+k}$ $\quad 1+k$

$$\rho = \lim_{k\to\infty} \frac{1}{1+(k+1)}$$

(15)

$$= \lim_{k\to\infty} \frac{k}{k+1} = \lim_{k\to\infty} \frac{k}{k(1+1/k)} = 1$$

Test fail.

* $\sum_{k=1}^{\infty} \frac{1}{2k-1}$ $\quad 2k-1$

$$\rho = \lim_{k\to\infty} \frac{1}{2(k+1)-1}$$

$$= \lim_{k\to\infty} \frac{2k-1}{2k+1} = \lim_{k\to\infty} \frac{2k(1-1/2k)}{2k(1+\frac{1}{2k})} = \underline{1}$$

Test fail.

We have $\quad 2k-1 < 2k.$

$$\frac{1}{2k-1} > \frac{1}{2k}$$

Take $\sum b_k = \sum \frac{1}{2k} = \frac{1}{2} \sum \frac{1}{k}$ diverges

$\therefore$ By comparison test $\sum_{k=1}^{\infty} \frac{1}{2k-1}$ also diverges

use the ratio test ~~the~~ to determine whether the series converges. If the test is inconclusive, then say so

$$* \sum_{K=1}^{\infty} \frac{99^K}{K!}$$

$$\rho = \lim_{K\to\infty} \frac{a_{K+1}}{a_K} = \lim_{K\to\infty} \frac{(99)^{K+1}}{(K+1)!} \frac{K!}{(99)^K}$$

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$$= \lim_{K\to\infty} \frac{99}{K+1} = 0 < 1$$

Hence the series is convergent by ratio test.

$$* \sum_{K=1}^{\infty} \frac{4^K}{K^2}$$

$$\rho = \lim_{K\to\infty} \frac{4^{K+1}}{(K+1)^2} \cdot \frac{K^2}{4^K}$$

$$= \lim_{K\to\infty} 4 \left(\frac{K^2}{K^2(1+\frac{1}{K})^2} \right)$$

$$= \lim_{K\to\infty} \frac{4}{(1+\frac{1}{K})^2} = 4 > 1$$

$\therefore$ Series diverges by ratio test.

$$* \sum_{K=1}^{\infty} \frac{K!}{K^{99}}$$

$$\rho = \lim_{K\to\infty} \frac{(K+1)!}{(K+1)^{99}} \cdot \frac{K^{99}}{K!}$$

$= \infty$, Hence divergent

$$* \sum_{K=1}^{\infty} \frac{K}{K^2+1}$$

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$$\rho = \lim_{K \to \infty} \frac{K+1}{(K+1)^2+1} \times \frac{K^2+1}{K}$$

$$= \lim_{K \to \infty} \frac{\cancel{K}(1+1/K)}{K^2+2K+2} \cdot \frac{K^2+1}{\cancel{K}}$$

$$= 1$$ Test fail

The root test

Let $\sum u_K$ be a series with positive terms and suppose that $\rho = \lim_{K \to \infty} \sqrt[K]{u_K} = \lim_{K \to \infty} (u_K)^{1/K}$

(a) If $\rho < 1$, the series converges

(b) If $\rho > 1$ or $\rho = +\infty$, the series diverges

(c) If $\rho = 1$, the series may converges or diverge so that another test must be tried.

[Try this test when u_K involve K^{th} power)

* use the root test to determine whether the series converges. ~~Of the test~~

* (1) $\sum_{K=1}^{\infty} \left(\frac{K}{100}\right)^K$

(34)

$$\rho = \lim_{k\to\infty}\left[\left(\frac{k}{100}\right)^k\right]^{1/k}$$

$$= \lim_{k\to\infty} \frac{k}{100}$$

$$= \infty.$$

$\therefore$ The Series is diverges by Root test

(2) $\displaystyle\sum_{k=1}^{\infty}\left(\frac{3k+2}{2k-1}\right)^k$

$$\rho = \lim_{k\to\infty}\left[\left(\frac{3k+2}{2k-1}\right)^k\right]^{1/k}$$

$$= \lim_{k\to\infty}\frac{3k+2}{2k-1} = \lim_{k\to\infty}\frac{k\left(3+\frac{2}{k}\right)}{k\left(2-\frac{1}{k}\right)} =$$

$\therefore$ the series is diverges.

(3) $\displaystyle\sum_{k=1}^{\infty}\frac{1}{(\ln(k+1))^k}$

$$\rho = \lim_{k\to\infty}\left[\frac{1}{(\ln(k+1))^k}\right]^{1/k}$$

$$= \lim_{k\to\infty}\frac{1}{\ln(k+1)}$$

$= 0 < 1$, the Series converges

(4) $\displaystyle\sum_{k=1}^{\infty}(1-e^{-k})^k$

$$\rho = \lim_{k\to\infty}\left((1-e^{-k})^k\right)^{1/k} = \lim_{k\to\infty} 1-e^{-k}$$

$= 1$, inconclusive

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Find the general term of the series and use the ratio test to show that the series converges

(19)

(1) $1 + \frac{1\cdot 2}{1\cdot 3} + \frac{1\cdot 2\cdot 3}{1\cdot 3\cdot 5} + \frac{1\cdot 2\cdot 3\cdot 4}{1\cdot 3\cdot 5\cdot 7} + \cdots$

General term is, $\frac{1\cdot 2\cdot 3\cdot 4\cdots k}{1\cdot 3\cdot 5\cdot 7\cdots (2k-1)}$

$$\sum_{k=1}^{\infty} \frac{1\cdot 2\cdots (k)}{1\cdot 3\cdot 5\cdots (2k-1)} = \sum_{k=1}^{\infty} \frac{k!}{1\cdot 3\cdots (2k-1)} \cdot \frac{2\cdot 4\cdot 6\cdot 8\cdots 2k}{2\cdot 4\cdot 6\cdot 8\cdots 2k}$$

$$= \sum_{k=1}^{\infty} \frac{k!\, 2\cdot 4\cdot 6\cdot 8\cdots 2k}{\cancel{2k}\, 1\cdot 2\cdot 3\cdot 4\cdots (2k-1)2k}$$

$$= \sum_{k=1}^{\infty} \frac{k!\, 2\cdot 4\cdot 6\cdot 8\cdots 2k}{(2k)!}$$

$$= \sum_{k=1}^{\infty} \frac{k!\, 2^k [1\cdot 2\cdot 3\cdots k]}{(2k)!}$$

$$= \sum_{k=1}^{\infty} \frac{(k!)^2\, 2^k}{(2k)!}$$

$n! = n(n-1)!$

$$\rho = \lim_{k\to\infty} \frac{a_{k+1}}{a_k}$$

$$= \lim_{k\to\infty} \frac{((k+1)!)^2\, 2^{k+1}}{(2(k+1))!} \times \frac{2k!}{(k!)^2\, 2^k}$$

$$= \lim_{k\to\infty} 2\cdot \left(\frac{(k+1)!}{k!}\right)^2 \cdot \frac{2k!}{(2k+2)!}$$

$$= \lim_{k\to\infty} 2(k+1)^2 \frac{2k!}{(2k+2)(2k+1)2k!}$$

(36)

$$\lim_{k\to\infty} \frac{2(k+1)^2}{(2k+2)(2k+1)} = \lim_{k\to\infty} \frac{2(k^2+2k+1)}{4k^2+6k+2}$$

$$= 2 \times \frac{1}{4}$$

$$= \frac{1}{2} < 1 \quad \text{converges}$$

(2)

use any method to determine whether the series converges.

(1) $\sum_{k=1}^{\infty} \frac{7\cos^2 k}{k!}$

we have $\cos^2 k \le 1$

$$\therefore \frac{7\cos^2 k}{k!} \le \frac{7}{k!}$$

consider the series $\sum_{k=1}^{\infty} \cancel{\frac{7\cos^2 k}{k!}} = \sum_{k=1}^{\infty} \frac{7}{k!} = \sum_{k=1}^{\infty} b_k$

$$\rho = \lim_{k\to\infty} \frac{b_{k+1}}{b_k} = \lim_{k\to\infty} \frac{7}{(k+1)!} \cdot \frac{k!}{7}$$

$$= \lim_{k\to\infty} \frac{1}{k+1}$$

$$= 0 < 1$$

$\sum b_k$ converges by ratio test

Hence by comparison Test $\sum a_k = \sum_{k=1}^{\infty} \frac{7\cos^2 k}{k!}$ also conv

* $\sum_{k=1}^{\infty} k^{50} e^{-k}$

$$\rho = \lim_{k\to\infty} \frac{a_{k+1}}{a_k}$$

$$= \lim_{k\to\infty} \frac{(k+1)^{50} e^{-(k+1)}}{k^{50} e^{-k}} = \lim_{k\to\infty} \left(\frac{k+1}{k}\right)^{50} e^{-1}$$

$$= \lim_{k\to\infty} \left(\frac{k(1+1/k)}{k}\right)^{50}$$

$$= \frac{1}{e} < 1$$

(37) 21

By ratio test the series is convergent.

$$* \sum_{K=1}^{\infty} \left[\frac{\pi(K+1)}{K^{K+1}}\right]^{K}$$

$$\rho = \lim_{K\to\infty} (a_K)^{1/K}$$

$$= \lim_{K\to\infty} \left(\left(\frac{\pi(K+1)}{K}\right)^{K} \right)^{1/K}$$

$$= \lim_{K\to\infty} \frac{\pi(K+1)}{K} \frac{1}{K^{1/K}} \qquad \lim_{K\to\infty} K^{1/K} = 1$$

$= \pi > 1 \quad \therefore$ Divergent Series

$$* \lim_{K\to 1} \sum_{K=1}^{\infty} \frac{1}{5K^2-2K}$$

$$K^2 \geq K$$
$$-K^2 \leq -K$$
$$-2K^2 \leq -2K$$
$$5K^2-2K^2 \leq 5K^2-2K$$
$$3K^2 \leq 5K^2-2K$$
$$\frac{1}{3K^2} \geq \frac{1}{5K^2-2K}$$
$$\sum_{K=1}^{\infty} \frac{1}{3K^2} \text{ cgt}$$

$$\therefore \sum_{K=1}^{\infty} \frac{1}{5K^2-2K} \text{ cgt.}$$

$$* \sum_{K=1}^{\infty} \frac{\tan^{-1} K}{K^2}$$

$$b_K = \frac{1}{K^2}. \qquad \sum_{K=1}^{\infty} \frac{1}{K^2} \text{ is cgt}$$

$$\lim_{K\to\infty} \frac{\tan^{-1} K}{K^2} \times K^2$$

$$\lim_{K\to\infty} \tan^{-1} K = \tan^{-1}\infty = \pi/2 \quad \text{finite} \cdot \text{cgt}$$

* $\sum_{k=1}^{\infty} \frac{2k^2+1}{2k^{8/3}-1}$

$2k^2+1 \geq 2k^2$

$2k^{8/3}-1 \leq 2k^{8/3}$

(22)

$\frac{2k^2+1}{2k^{8/3}-1} \geq \frac{2k^2}{2k^{8/3}}$

$\frac{2k^2+1}{2k^{8/3}-1} \geq \frac{1}{k^{2/3}}$ is divergent

Hence $\sum_{k=1}^{\infty} \frac{2k^2+1}{2k^{8/3}-1}$ is divergent

Note

Let Σa_k and Σb_k be series with +ve terms.

(a) If $\lim_{k\to\infty} (a_k/b_k) = 0$ and Σb_k converges, then Σa_k

(b) If $\lim_{k\to\infty} (a_k/b_k) = \infty$ and Σb_k diverges, then Σa_k

* $\sum_{k=1}^{\infty} \frac{\ln k}{k}$ $\quad a_n = \frac{\ln k}{k}$

let $b_k = \frac{1}{k}$

$\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k}$ which is dgt $(P=1)$

$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\frac{\ln k}{k}}{1/k} = \lim_{k\to\infty} \frac{\ln k}{k} \times \frac{k}{1}$

$= \lim_{k\to\infty} \ln k = \underline{\underline{\infty}}$

$\therefore \Sigma b_k$ diverges, then $\sum_{k=1}^{\infty} \frac{\ln k}{k}$ is dgt.

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$2-\frac{8}{3}$
$-2/3$

$$\sum_{k=1}^{\infty} \frac{2k^2+1}{2k^{8/3}-1}$$

$$a_k = \frac{2k^2+1}{2\cancel{8}k^{8/3}-1} = \frac{\cancel{2}k^2\left[1+\frac{1}{2k^2}\right]}{\cancel{2}k^{8/3}\left[1-\frac{1}{2k^{8/3}}\right]} = \frac{1}{k^{2/3}}\frac{\left[1+\frac{1}{2k^2}\right]}{\left[1-\frac{1}{2k^{8/3}}\right]}$$

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$$b_k = \frac{1}{k^{2/3}}$$

$\sum_{k=1}^{\infty} b_k$ cgs by p series since $p = 2/3 < 1$

$$\therefore \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\cancel{1/k^{2/3}}\left[1+1/2k^2\right]}{\cancel{k^{2/3}}\left[1-1/2k^{8/3}\right]}$$

$= 1$ finite and >0

Hence the given series is ~~convergent~~ divergent by limit Comparison test.

Alternating Series

A series in which ~~those~~ the terms are alternate positive and negative is called an Alternating Series

Eg $\sum_{k=1}^{\infty} (-1)^{k+1}\frac{1}{k} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}- \cdots$

$\sum_{k=1}^{\infty} (-1)^{k}\frac{1}{k} = -1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+ \cdots$

In general, an alternating series has one of the following two forms;

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$$\sum_{k=1}^{\infty} (-1)^{k+1} a_k = a_1 - a_2 + a_3 - a_4 + \cdots$$

$$\sum_{k=1}^{\infty} (-1)^{k} a_k = -a_1 + a_2 - a_3 + a_4 - \cdots$$

24 Where a_k's are assumed to be positive in both case

Absolute Convergence

A Series $\sum_{k=1}^{\infty} u_k = u_1 + u_2 + u_3 + \cdots + u_k + \cdots$ is

Said to converge absolutely if the series of absolut

$$\sum_{k=1}^{\infty} |u_k| = |u_1| + |u_2| + |u_3| + \cdots + |u_k| + \cdots$$

converges and is said to diverge absolutely if the series of absolute values diverges.

Eg:- * Determine whether the following series converg absolutely.

(a) $1 - \frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} - \frac{1}{2^5} \cdots\cdots$

$$\sum_{k=1}^{\infty} |u_k| = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$$ which is a

~~cgt~~ convergent geometric series. Hence the given series is convergent absolutely.

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$

$$\sum_{k=1}^{\infty} |u_k| = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

which is harmonic Series. The absolute value is a divergent Harmonic Series. Hence it is diverges absolutely.

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Theorem

If the series $\sum_{k=1}^{\infty} |u_k| = |u_1| + |u_2| + |u_3| + \cdots + |u_k| + \cdots$

Converges, then so does the Series.

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + u_3 + \cdots + u_k + \cdots$$

Conditional convergence

An infinite Series $\sum a_n$ is convergent conditionally if $\sum a_n$ is convergent but its absolute ~~value~~ Series $|\sum a_n|$ is divergent.

Eg:- Consider the Series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots (-1)^{k+1} \frac{1}{k} + \cdots \longrightarrow (1)$$

which is a Conditionally convergent Series. Because its absolute value is the divergent Harmonic Series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{k} + \cdots \longrightarrow (2)$$

However, Series (1) Converges, Since it is the alternating

Harmonic series, and Series (2) diverges, since it is constant times the divergent Harmonic Series.

Thus (1) is a conditionally convergent series.

(26)

Problems

(1) Determine whether the Series converges absolutely or converges conditionally.

$$\sum_{K=1}^{\infty} \frac{\cos K}{K^2}$$

We have $|\cos K| \leq 1$

$$\frac{|\cos K|}{K^2} \leq \frac{1}{K^2}.$$

But $\sum \frac{1}{K^2}$ is a convergent p series ($p=2$), So the Series of ~~absolute~~ absolute values converge by the comparison test. Thus the given Series converges absolutely and hence converges.

(2) $$\sum_{K=1}^{\infty} (-1)^{K+1} \frac{K+3}{K(K+1)}$$

Given Series is ~~Abs~~ absolutely convergent if,

$$\sum_{K=1}^{\infty} \left| (-1)^{K+1} \frac{K+3}{K(K+1)} \right|$$ is convergent

(43)

$$\left|\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k+3}{k(k+1)}\right| = \sum_{k=1}^{\infty} \frac{k+3}{k(k+1)} \rightarrow (1)$$

Let $b_k = \frac{1}{k}$, then $\sum b_k = \sum \frac{1}{k}$ which is a divergent Series with $p = 1$. and $a_k = \sum_{k=1}^{\infty} \frac{k+3}{k(k+1)}$

(27)

$$\rho = \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\frac{k+3}{\cancel{k}(k+1)}}{\cancel{k}}$$

$$= \lim_{k\to\infty} \frac{\cancel{k}(1+\frac{3}{k})}{\cancel{k}(1+\frac{1}{k})} = 1$$

Here ρ is finite and $\rho > 0$. Hence $\sum_{k=1}^{\infty} \frac{k+3}{k(k+1)}$ is convergent or divergent. Since $\sum b_k$ is divergence, $\sum_{k=1}^{\infty} \frac{k+3}{k(k+1)}$ is divergent.

From (1) $\sum_{k=1}^{\infty} |a_k|$ is divergent.

or $\sum a_k$ is absolutely divergent.

Ratio Test for absolute Convergence

Let $\sum u_k$ be a series with non zero terms and Suppose that $\rho = \lim_{k\to\infty} \frac{|u_{k+1}|}{|u_k|}$

(a) if $\rho < 1$ then the Series $\sum u_k$ converges absolutely and $\therefore$ converges.

(b) If $\rho > 1$ or $\rho = \infty$ then the Series $\sum u_k$ diverges

(c) if $\rho = 1$, no conclusion about convergence.

Problems (59) Determine whether the Series is converge or diverge

(a) $\sum_{k=1}^{\infty} \frac{(-1)^k 2^k}{k!}$

Taking the absolute value of the general term u_k, we obtain $|u_k| = \left|\frac{(-1)^k 2^k}{k!}\right| = \frac{2^k}{k!}$

(26)

Thus

$$\rho = \lim_{k\to\infty} \frac{|u_{k+1}|}{|u_k|} = \lim_{k\to\infty} \frac{2^{k+1}}{(k+1)!} \times \frac{k!}{2^k}$$

$$= 2\lim_{k\to\infty} \frac{1}{k+1}$$

$$= 0 < 1$$

Since $\rho < 1$ which implies that the Series converges hence absolutely converges. And therefore Converges.

(b) $\sum_{k=1}^{\infty} \frac{(-1)^k (2k-1)!}{3^k}$

$$|u_k| = \left|\frac{(-1)^k (2k-1)!}{3^k}\right| = \frac{(2k-1)!}{3^k}$$

$$\rho = \lim_{k\to\infty} \frac{|u_{k+1}|}{|u_k|} = \lim_{k\to\infty} \frac{[2(k+1)-1]! \times 3^k}{3^{k+1} (2k-1)!}$$

$$= \lim_{k\to\infty} \frac{(2k+1)!}{3} \cdot \frac{1}{(2k-1)!}$$

$$= \lim_{k\to\infty} \frac{(2k+1)\,2k\,(2k-1)!}{3} \cdot \frac{1}{(2k-1)!}$$

$$= \frac{1}{3} \lim_{k\to\infty} 2k(2k+1) = \infty$$

which implies that the series diverges.

(c) $\sum_{k=1}^{\infty} \frac{(-1)^k k^5}{e^k}$

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$$|u_k| = \left| \cancel{\sum_{k=1}^{\infty}} (-1)^k \frac{k^5}{e^k} \right| = \frac{k^5}{e^k}$$

Thus $\rho = \lim_{k\to\infty} \frac{|u_{k+1}|}{|u_k|} = \lim_{k\to\infty} \frac{(k+1)^5}{e^{k+1}} \times \frac{e^k}{k^5}$

$$= \lim_{k\to\infty} \frac{k^5\left(1+\frac{1}{k}\right)^5 \times}{k^5 e}$$

$$= \frac{1}{e} \lim_{k\to\infty} \left(1+\frac{1}{k}\right)^5$$

$$= \frac{1}{e} < 1$$

Since $\rho < 1$ which implies that the series converges hence absolutely converges. And therefore converges.

(d) $\sum_{k=1}^{\infty} \frac{k \cos k\pi}{k^2+1}$

We have $|\cos k\pi| = |(-1)^k| = 1$

$$\left| \cancel{\sum_{k=1}^{\infty} \frac{k\cos k\pi}{k^2+1}} \right| = \cancel{\sum_{k=}}$$

$$\left| \sum_{k=1}^{\infty} \frac{k \cos k\pi}{k^2+1} \right| = \sum_{k=1}^{\infty} \frac{k}{k^2+1} \rightarrow (1) \text{ Let } a_k = \frac{k}{k^2+1}$$

choose $b_k = \frac{1}{k}$

Now $\sum b_k$ is divergent with p series $p = 1$

$$\rho = \lim_{k \to \infty} \frac{a_k}{b_k} = \lim_{k \to \infty} \frac{\frac{k}{k^2+1}}{1/k} \times 1$$

(2)

$$= \lim_{k \to \infty} \frac{k}{k^2+1} \times k$$

$$= \lim_{k \to \infty} \frac{k}{\cancel{k^2}\left(1+\frac{1}{k^2}\right)} \times \cancel{k}$$

$$= \lim_{k \to \infty} \frac{\cancel{k}}{\cancel{k}\left(1+\frac{1}{k^2}\right)}$$

$$= \lim_{k \to \infty} \frac{1}{1+\frac{1}{k^2}} = 1 \text{ finite}$$

Hence the given series is cgt or divergent together. Since b_k is divergent

$\therefore \sum_{k=1}^{\infty} \left| \frac{k \cos k\pi}{k^2+1} \right|$ is divergent

$\therefore$ Given series is not absolutely convergent.

H.W (1) $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{3^k}{k^2}$

$$|a_k| = \left| (-1)^{k+1} \frac{3^k}{k^2} \right| = \frac{3^k}{k^2}$$

Thus $\rho = \lim_{k \to \infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k \to \infty} \frac{3^{k+1}}{(k+1)^2} \times \frac{k^2}{3^k}$

$$= 3 \lim_{k \to \infty} \frac{k^2}{k^2\left[1+\frac{1}{k^2}\right]} = 3 > 1,$$ wh

B. finite, > 0

$$\text{Thus } \rho = \lim_{k\to\infty} \frac{|u_{k+1}|}{|u_k|} = \lim_{k\to\infty} \frac{3^{k+1}}{(k+1)^2} \times \frac{k^2}{3^k}$$

$$= \lim_{k\to\infty} \frac{3 \cancel{k^2}}{\cancel{k^2}\left(1+\frac{1}{k^2}\right)}$$

(31)

$$= 3 > 1$$

∴ Thus the given series is divergent. Hence not absolutely convergent.

Leibnitz's Test on Alternating Series

The alternating series $\sum(-1)^{n-1} u_n = u_1 - u_2 + u_3 - u_4 \dots$ cgs if (i) $u_n > u_{n+1}$ $\forall n$ and (ii) $\lim_{k\to\infty} u_n = 0$.

Pbms

1) Examine the convergence of the series $1 - 1/2 + 1/3 - 1/4 \dots$

$u_n = 1/n$ $\quad u_{n+1} = 1/(n+1)$ $\quad$ (1) $u_n > u_{n+1}$ $\forall n$.

(2) $\lim_{n\to\infty} u_n = 0$

∴ By Leibnitz's test the series cgt

2) Examine the cgence of the series

$$2 - 3/2 + 4/3 - 5/4 \dots$$

$u_n = \frac{n+1}{n}$ $\quad u_{n+1} = \frac{n+2}{n+1}$.

$$u_n - u_{n+1} = \frac{n+1}{n} - \frac{n+2}{n+1} = \frac{(n+1)^2 - n(n+2)}{n(n+1)} = \frac{1}{n(n+1)} > 0 \ \forall n$$

(1)

∴ $u_n > u_{n+1}$. (ii) $\lim_{n\to\infty} u_n = 1 \neq 0$. Series not cgt

3) $\frac{1}{2^3} - \frac{1}{3^3}(1+2) + \frac{1}{4^3}(1+2+3) - \cdots$

$$u_n = \frac{1}{(n+1)^3}[1+2+\cdots+n] = \frac{n(n+1)}{2(n+1)^3}$$

$$= \frac{n}{2(n+1)^2}$$

(3)

$$u_{n+1} = \frac{n+1}{2(n+2)^2}.$$

$$u_n - u_{n+1} = \frac{n}{2(n+1)^2} - \frac{n+1}{2(n+2)^2}$$

$$= \frac{n(n+2)^2 - (n+1)(n+1)^2}{2(n+1)^2(n+2)^2}$$

$$= \frac{n[n^2+4n+4] - (n+1)[n^2+2 \cdots}{2(n+1)^2(n+2)^2}$$

$$= \frac{n^2+n-1}{2(n+1)^2(n+2)^2} > 0 \quad \forall n$$

$u_n > u_{n+1} \quad \forall n$

(ii) $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{2(n+1)^2} = 0$

$\therefore$ By Leibnitz's Test Series cgs copy 4 copy

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Module III

Fourier Series

Periodic function

A function $f(x)$ which satisfies the relation $f(x+T) = f(x)$ for all real x and some fixed T is called periodic function. The smallest positive number T for which this relation holds is called the period of $f(x)$.

If T is a period of $f(x)$ then $f(x) = f(x+T) = f(x+2T) = \dots = f(x+nT) = \dots$

Eg: $\cos x$, $\cos 2x$, $\cos 3x$ are periodic function with periods 2π, π, $\frac{2\pi}{3}$ respectively.

$\sin x$, $\cos x$, $\operatorname{cosec} x$, & $\sec x$ are periodic functions with period 2π.

$\tan x$ & $\cot x$ are periodic functions with period π.

The functions $\sin nx$ & $\cos nx$ are periodic with period $\frac{2\pi}{n}$.

Even and odd functions

A function $f(x)$ is said to be even if $f(-x) = f(x)$

Eg: x^2, $\cos x$, $\sin^2 x$, $|x|$

The graph of an even function is symmetrical about y axis.

A function $f(x)$ is said to be odd if $f(-x) = -f(x)$

Eg: x^3, $\sin x$, $\tan^3 x$ are odd functions.

The graph of an odd function is symmetric about the origin.

The even fun $\times$ even fun = even fun.

odd " $\times$ odd " = even "

odd " $\times$ even " = odd function.

$$\int_{-c}^{c} f(x)\,dx = 0 \quad \text{if } f(x) \text{ is odd.}$$

$$\int_{-c}^{c} f(x)\,dx = 2\int_{0}^{c} f(x)\,dx, \quad \text{if } f(x) \text{ is even.}$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]$$

$$\int_{0}^{\infty} e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2} \quad \text{also} \quad \int_{0}^{\infty} e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2}$$

$\sin(A+B) = \sin A\cos B + \cos A\sin B$

$\sin(A-B) = \sin A\cos B - \cos A\sin B$

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\cos(A-B) = \cos A\cos B + \sin A\sin B$

$\sin A\cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$

$\cos A\sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$

$\cos A\cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$

$\sin A\sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$

Orthogonality Property of Sine & Cosine function

$$\int_{t_0}^{t_0+T} \sin(m\omega t)\sin(n\omega t)\,dt = 0 \qquad m \neq n,\ m=n=0.$$

$$= T/2 \qquad m = n \neq 0.$$

$$\int_{t_0}^{t_0+T} \cos(m\omega t)\cos(n\omega t)\,dt = 0 \qquad m \neq n$$

$$= T/2 \qquad m = n \neq 0$$

$$= T. \qquad m = n = 0$$

Fourier Series representation of 2π periodic function in the interval $(-\pi, \pi)$

Suppose that $f(x)$ is a periodic function of period 2π and that can be represented by the trigonometric series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx.$$

Where a_0, a_n, b_n are Fourier co-efficients

Where

$$\left.\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \end{aligned}\right\} \longrightarrow (1)$$

These formulas are called Euler's formulas.

If the interval $(0, 2\pi)$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx.$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx.$$

If $f(x)$ is odd in $(-\pi, \pi)$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx \quad 0$$

$[\cancel{f(x)} \rightarrow$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx.$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx\,dx \quad \text{Where.}$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx.$$

$f(x)$ even

$$a_0 = 2/\pi \int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$b_n = 0$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

Pbms

1 Find the Fourier Series of the function $f(x) = x$ in the interval $-\pi < x < \pi$.

Ans Fourier Series representation of $f(x)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx.$$

Here $f(x) = x$ $\quad f(-x) = -x = -f(x)$

$f(x)$ odd.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \cancel{dx} \qquad [\because a_0 = a_n = 0.$$

$$b_n = 2/\pi \int_0^{\pi} f(x) \sin nx\,dx$$

$$= 2/\pi \int_0^{\pi} x \sin nx\,dx$$

$$= \frac{2}{\pi}\left[x \times \frac{-\cos nx}{n} - 1 \times -\frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$= 2/\pi \left[-\pi \frac{\cos n\pi}{n} + 0 - (0-0)\right]$$

$$\therefore \frac{-2\cos n\pi}{n} = -\frac{2}{n}(-1)^n \quad \underline{\underline{2/n\,(-1)^{n+1}}}$$

$$\therefore f(x) = \sum_{n=1}^{\infty} 2/n\,(-1)^{n+1} \sin nx \cancel{dx}.$$

$$= [\,2\sin x - 2/2 \sin 2x + 2/3 \sin 3x - 2/4 \sin 4x \ldots$$

$$= 2\left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} \cdots\right)$$

3. Find the Fourier Series of $f(x)$ given by
$$f(x) = \begin{cases} -k & -\pi < x < 0 \\ k & 0 < x < \pi \end{cases}$$ Hence deduce that $\pi/4 = 1 - 1/3 + 1/5$

A: Fourier Series of $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -k\,dx + \int_{0}^{\pi} k\,dx\right] = \frac{1}{\pi}\left[[-kx]_{-\pi}^{0} + [kx]_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}(-k\pi + k\pi) = 0$$

$$\boxed{a_0 = 0}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -k\cos nx\,dx + \int_{0}^{\pi} k\cos nx\,dx\right]\Big]$$

$$= \frac{1}{\pi}\left[-kx\frac{\sin nx}{n}\Big]_{-\pi}^{0} + k\frac{\sin nx}{n}\Big]_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}[0.] \qquad \boxed{a_n = 0}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx..$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -k\sin nx\,dx + \int_{0}^{\pi} k\sin nx\,dx.\right]$$

$$= \frac{1}{\pi}\left[\frac{k\cos nx}{n}\Big]_{-\pi}^{0} - \frac{k\cos nx}{n}.\Big]_{0}^{\pi}\right]$$

$$= \frac{1}{k}\left[\left(\frac{k}{n} - \frac{k}{n}\cos n\pi\right) - \left(\frac{k}{n}\cos n\pi - \frac{k}{n}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{2k}{n} - \frac{2k}{n}(-1)^n\right] = \frac{2k}{n\pi}(1 - (-1)^{n})$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx = \sum_{n=1}^{\infty} \frac{2k}{n\pi}(1 - (-1)^n)\sin nx$$

$$\frac{2k}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(1-(-1)^n)\sin nx$$

$$=\frac{2k}{\pi}\left[\frac{1}{1}2\sin x+0+\frac{1}{3}\times 2\sin 3x+0+\frac{1}{5}\ldots\right.$$

$$f(x)=\frac{4k}{\pi}\left[\sin x+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}+\cdots\right]$$

$x=\pi/2$ $$K=\frac{4k}{\pi}\left[\sin\pi/2+\frac{\sin\frac{3\pi}{2}}{3}+\frac{\sin 5\pi/2}{5}+\cdots\right]$$

$$\pi/4=\left[1+-1/3+1/5\cdots\right]$$

$$\pi/4=1-1/3+1/5\cdots$$

$$\boxed{\sin n\pi=0 \qquad \cos n\pi=(-1)^n}$$

Sup

3. Expand in a Fourier series the function $f(x)=e^x$ in the interval $0<x<2\pi$.

A: $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+b_n\sin nx$. [function neither even nor odd]

$$a_0=\frac{1}{\pi}\int_0^{2\pi}f(x)\,dx$$

$$=\frac{1}{\pi}\int_0^{2\pi}e^x\,dx=\frac{1}{\pi}(e^x)_0^{2\pi}=\frac{1}{\pi}(e^{2\pi}-1)$$

$$\boxed{a_0=\frac{e^{2\pi}-1}{\pi}}$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nx\,dx=\frac{1}{\pi}\int_0^{2\pi}e^x\cos nx\,dx$$

$$=\frac{1}{\pi}\left.\frac{e^x}{1+n^2}(\cos nx+n\sin nx)\right]_0^{2\pi}$$

$$=\frac{1}{\pi}\left[\frac{e^{2\pi}}{n^2+1}(\cos 2n\pi+n\sin 2n\pi)-\frac{1}{n^2+1}(\cos 0+n\sin 0)\right]$$

$$\boxed{a_n=\frac{1}{\pi(n^2+1)}\left[e^{2\pi}-1\right]}$$

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$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{2\pi} e^{x}\sin nx\,dx\right] = \frac{1}{\pi}\left[\frac{e^{x}}{n^2+1}\left[\sin nx - n\cos nx\right]\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{2\pi}}{n^2+1}\left[\left(\sin 2n\pi - n\cos 2n\pi\right)\right] - \frac{1}{n^2+1}\left[\sin 0 - n\cos 0\right]\right]$$

$$= \frac{1}{\pi(n^2+1)}\left[-ne^{2\pi} + n\right] = \frac{n}{\pi(n^2+1)}\left(1 - e^{2\pi}\right)$$

$$\boxed{b_n = \frac{n}{\pi(n^2+1)}\left(1 - e^{2\pi}\right)}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + b_n\sin nx$$

$$= \frac{1}{2\pi}(e^{2\pi}-1) + \sum_{n=1}^{\infty}\frac{1}{\pi(n^2+1)}(e^{2\pi}-1)\cos nx + \frac{n(1-e^{2\pi})}{\pi(n^2+1)}\sin nx.$$

$$= \frac{1}{2\pi}(e^{2\pi}-1) + \frac{e^{2\pi}-1}{\pi}\left[\sum_{n=1}^{\infty}\frac{\cos nx}{n^2+1} - \sum_{n=1}^{\infty}\frac{n}{n^2+1}\sin nx\right]$$

$$= \frac{1}{2\pi}(e^{2\pi}-1) + \frac{e^{2\pi}-1}{\pi}\left[\left[\frac{\cos x}{1^2+1} + \frac{\cos 2x}{2^2+1} + \cdots\right] - \left[\frac{\sin x}{1^2+1} + \frac{2}{2^2+1}\sin 2x + \cdots\right]\right]$$

$$\frac{1}{2\pi}(e^{2\pi}-1) + \frac{e^{2\pi}-1}{\pi}\left[\left(\frac{\cos x}{2} + \frac{\cos 2x}{5} + \cdots\right) - \left(\frac{\sin x}{2} + \frac{2\sin x}{5}\cdots\right)\right]$$

$$= \frac{e^{2\pi}-1}{\pi}\left[\frac{1}{2} + \left(\frac{\cos 2x}{2} + \frac{\cos 2x}{5}\cdots\right) - \left(\frac{\sin x}{2} + \frac{2\sin 2x}{5} + \cdots\right)\right]$$

4. Expand in Fourier series the function $f(x) = |x|$ in the interval $-\pi < x < \pi$. Deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx.$$

$f(x) = |x|$ $\quad f(-x) = |-x| = |x| = f(x)$ even function

$\therefore b_n = 0.$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx. = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{2}{\pi} \cdot$$

$= \underline{\underline{\pi}}$

$$\boxed{a_0 = \pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx. = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx$$

$$= \frac{2}{\pi}\left[x \frac{\sin nx}{n} - 1 \cdot \frac{-\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\pi \frac{\sin n\pi}{n} + \frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2}\right]$$

$$= \frac{2}{\pi}\left(\frac{(-1)^n - 1}{n^2}\right) \qquad \boxed{a_n = \frac{2}{\pi n^2}\left[(-1)^n - 1\right]}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

$$= \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}\left((-1)^n - 1\right)\cos nx$$

$$= \frac{\pi}{2} + \frac{2}{\pi}\left[\frac{-2\cos x}{1^2} + \frac{-2\cos 3x}{3^2} - \frac{2\cos 5x}{5^2}\right]$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \cdots\right]$$

put $x = 0.$ $\quad f(x) = 0.$

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos 0}{1^2} + \frac{\cos 0}{3^2} + \frac{\cos 0}{5^2} + \cdots\right]$$

$$\frac{\pi}{2} = \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

Expand $f(x) = x\sin x$, $-\pi < x < \pi$ as a Fourier Series. Deduce that $\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \frac{1}{7\cdot 9} = \frac{\pi - 2}{4}$

$f(x) = x\sin x$,

$f(-x) = -x\sin(-x) = x\sin x = f(x)$ even function

$b_n = 0$.

$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin x\,dx$.

$= \frac{2}{\pi}\Big[x \times -\cos x - \cdot -\sin x\Big]_0^{\pi} = \frac{2}{\pi}\Big[-\pi\cos\pi + 0 + (\sin\pi - \sin 0)\Big]$

$= \frac{2}{\pi} \times -\pi \times -1 = \underline{2}$

$\boxed{a_0 = 2}$

$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$.

$= \frac{1}{\pi}\int_0^{\pi} x(2\sin x\cos nx)\,dx$.

$= \frac{1}{\pi}\int_0^{\pi} x(2\cos nx\sin x)\,dx$

$= \frac{1}{\pi}\int_0^{\pi} x\big[\sin(n+1)x - \sin(n-1)x\big]\,dx$

$= \frac{1}{\pi}\Big[\int_0^{\pi} x(\sin(n+1)x\,dx - \int_0^{\pi} x\sin(n-1)x\,dx\Big]$

$= \frac{1}{\pi}\Big[\Big[x \times \frac{-\cos(n+1)x}{n+1} - 1 \times \frac{-\sin(n+1)x}{(n+1)^2}\Big]_0^{\pi} - \Big[x \times \frac{-\cos(n-1)x}{(n-1)} - 1 \times \frac{-\sin(n-1)x}{(n-1)^2}\Big]_0^{\pi}$

$= \frac{1}{\pi}\Big[\Big(-\pi\frac{\cos(n+1)\pi}{n+1} + \frac{\sin(n+1)\pi}{(n+1)^2}\Big) - \Big[\pi\frac{\cos(n-1)\pi}{(n-1)} + \frac{\sin(n-1)\pi}{(n-1)^2}\Big]$

$= \frac{1}{\pi}\Big[\pi\Big[\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1}\Big]\Big]$

$= -\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1}$ $\quad n \neq 1$

If $n=1$ $\quad a_1 = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos x \, dx$

$= \frac{1}{\pi}\int_0^{\pi} x \sin 2x \, dx$

$= \frac{1}{\pi}\left[x \times -\frac{\cos 2x}{2} - 1 \times -\frac{\sin 2x}{4} \right]_0^{\pi}$

$= \frac{1}{\pi}\left[\pi \times -\frac{\cos 2\pi}{2} \right] = -\frac{\cos 2\pi}{2} = -\frac{1}{2}$

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \, dx$

$= 1 + -\frac{1}{2}\cos x + \sum_{n=2}^{\infty}\left[\frac{\cos(n-1)\pi}{n-1} - \frac{\cos(n+1)\pi}{n+1} \right] \cos nx$

$= 1 - \frac{1}{2}\cos x + \left[\left(\frac{\cos \pi}{2-1} - \frac{\cos 3\pi}{2+1} \right) \cos 2x + \left(\frac{\cos 2\pi}{3-1} - \frac{\cos 4\pi}{3+1} \right) \cos 3x \right.$

$\left. + \left(\frac{\cos 3\pi}{4-1} - \frac{\cos 5\pi}{4+1} \right) \cos 4x \cdots \right]$

$= 1 - \frac{1}{2}\cos x + \left[\left(-\frac{1}{2-1} + \frac{1}{2+1} \right) \cos 2x + \left(\frac{1}{3-1} - \frac{1}{3+1} \right) \cos 3x + \right.$

$\left. \left(-\frac{1}{4-1} + \frac{1}{4+1} \right) \cos 4x + \left(\frac{1}{5-1} - \frac{1}{5+1} \right) \cos 5x - \right]$

$= 1 - \frac{1}{2}\cos x + \left[-\frac{2}{2^2-1}\cos 2x + \frac{2}{3^2-1}\cos 3x - \frac{2}{4^2-1}\cos 4x + \frac{2}{5^2-1}\cos 5x \cdots \right]$

$-\frac{1}{n-1} + \frac{1}{n+1}$

$= 1 - \frac{1}{2}\cos x - 2\left[\frac{\cos 2x}{3} - \frac{\cos 3x}{8} + \frac{\cos 4x}{15} - \frac{\cos 5x}{24} + \right]$

$-\frac{(n+1)+(n-1)}{n^2-1}$

$-\frac{2n}{n^2-1}$

$x = \pi/2 \quad f(x) = \pi/2 \sin \pi/2 = \pi/2$

$\frac{\pi}{2} = 1 - \frac{1}{2}\cos \pi/2 - 2\left[\frac{\cos \pi}{3} - \frac{\cos \frac{3\pi}{2}}{8} + \frac{\cos 2\pi}{15} - \frac{\cos \frac{5\pi}{2}}{24} \right]$

$= 1 - 2\left[-\frac{1}{3} + \frac{1}{15} \cdots \right]$

$\frac{\pi}{2} - 1 = 2\left(\frac{1}{3} - \frac{1}{15} + \frac{1}{35} \cdots \right) \quad = \frac{\pi - 2}{4} = \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} \cdots$

Find a Fourier series to represent $x-x^2$ from $x=-\pi$ to $x=\pi$. Hence deduce that $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}\cdots=\frac{\pi^2}{12}$.

$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos nx + b_n \sin nx$.

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx.$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\,dx = \frac{1}{\pi}\left(\frac{x^2}{2}-\frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$=\frac{1}{\pi}\left(\left(\frac{\pi^2}{2}-\frac{\pi^3}{3}\right)-\left(\frac{\pi^2}{2}-\left(-\frac{\pi^3}{3}\right)\right)\right)$$

$$=\frac{1}{\pi}\times-\frac{2\pi^3}{3} = -\frac{2}{3}\pi^2 \qquad \boxed{a_0=-\frac{2}{3}\pi^2}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\cos nx\,dx$$

$$=\frac{1}{\pi}\left\{(x-x^2)\times\frac{\sin nx}{n}-(1-2x)\times-\frac{\cos nx}{n^2}+-2x-\frac{\sin nx}{n^3}\right\}_{-\pi}^{\pi}$$

$$=\frac{1}{\pi}\left\{\left[(\pi-\pi^2)\frac{\sin n\pi}{n}+(1-2\pi)\frac{\cos n\pi}{n^2}+\frac{2\sin n\pi}{n^3}\right]-\left[(-\pi-\pi^2)\frac{\sin n(-\pi)}{n}+(1+2\pi)\frac{\cos n(-\pi)}{n^2}+\frac{2\sin n(-\pi)}{n^3}\right]\right\}$$

$$=\frac{1}{\pi}\left\{\frac{(1-2\pi)(-1)^n}{n^2}-\frac{(1+2\pi)(-1)^n}{n^2}\right\}$$

$$\frac{1}{\pi}\;\frac{-4\pi(-1)^n}{n^2} = -\frac{4}{n^2}(-1)^n \qquad \boxed{a_n=\frac{-4}{n^2}(-1)^n}$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\sin nx\,dx$$

$$=\frac{1}{\pi}\left\{(x-x^2)\times-\frac{\cos nx}{n}-(1-2x)\times-\frac{\sin nx}{n^2}+-2x\frac{\cos nx}{n^3}\right]_{-\pi}^{\pi}$$

$$\frac{1}{\pi}\left\{ \frac{-(\pi-\pi^2)\cos n\pi}{n} - \frac{2\cos n\pi}{n^3} - \left(-\frac{(\pi-\pi^2)\cos n\pi}{n} - \frac{2\cos n\pi}{n^3}\right]\right.$$

$$= \frac{1}{\pi}\left[-\frac{2\pi\cos n\pi}{n}\right] = -\frac{2}{n}(-1)^n. \quad \boxed{b_n = \frac{-2}{n}(-1)^n}$$

$$f(x) = x - x^2 = -\frac{1}{3}\pi^2 + \sum_{n=1}^{\infty} -\frac{4}{n^2}(-1)^n\cos nx - \frac{2}{n}(-1)^n \sin nx.$$

$$= -\frac{1}{3}\pi^2 - 4\sum_{n=1}^{\infty}\frac{(-1)^n\cos nx}{n^2} - 2\sum_{n=1}^{\infty}\frac{(-1)^n \sin nx}{n}.$$

$$= -\frac{1}{3}\pi^2 - 4\left[-\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2}\cdots\right]$$

$$-2\left[-\frac{\sin x}{1} + \frac{\sin 2x}{2} - \frac{\sin 3x}{3}\cdots\right]$$

$$f(x). = -\frac{\pi^2}{3} + 4\left[\frac{\cos x}{1} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2}\cdots\right]$$

$$+2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3}\cdots\right]$$

$x=0$

$$0 - 0^2 = -\frac{\pi^2}{3} + 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2}\cdots\right]$$

$$+2\left[0 - 0 + 0 \cdots\right]$$

$$0 = -\frac{\pi^2}{3} + 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2}\cdots\right]$$

$$\frac{\pi^2}{3} = 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

$$\boxed{\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2}\cdots}$$

7. Obtain the fourier series for the function $f(x) = x^2$ $-\pi < x < \pi$. Hence S.T

1) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \pi^2/6$

2) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \cdots = \pi^2/12$

3) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \pi^2/8$.

Ans $f(x) = x^2 \rightarrow$ even fun $\quad b_n = 0$.

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$.

$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$

$= \frac{2}{\pi}\int_0^{\pi} x^2\,dx = 2/\pi\left(x^3/3\right)_0^{\pi} = 2/\pi \cdot \pi^3/3 = \frac{2}{3}\pi^2 \quad \boxed{a_0 = \frac{2\pi^2}{3}}$

$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx$.

$= \frac{2}{\pi}\left[x^2 \times \frac{\sin nx}{n} - 2x \times \frac{-\cos nx}{n^2} + 2 \times \frac{-\sin nx}{n^3}\right]_0^{\pi}$

$= \frac{2}{\pi}\left[2\pi \frac{(-1)^n}{n^2}\right] = \frac{4(-1)^n}{n^2}$. $\quad \boxed{a_n = \frac{4(-1)^n}{n^2}}$

$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos nx$.

$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos nx$

$= \frac{\pi^2}{3} + 4\left[-\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} \cdots\right.$

$x^2 = \frac{\pi^2}{3} - 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} \cdots\right]$

Put $x = \pi$

$\pi^2 = \frac{\pi^2}{3} - 4\left[\frac{\cos\pi}{1^2} - \frac{\cos 2\pi}{2^2} + \frac{\cos 3\pi}{3^2} \cdots\right]$

$\pi^2 = \frac{\pi^2}{3} - 4\left[-\frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} \cdots\right]$

$\frac{2\pi^2}{3} = 4\left(\frac{1}{1^2} + \frac{1}{2^2} + 1/3^2 + \cdots\right) \Rightarrow \frac{1}{1^2} + \frac{1}{2^2} + \cdots = \frac{2\pi^2}{12} = \pi^2/6$

put $x = 0$.

$$0 = \frac{\pi^2}{3} - 4\left[\frac{\cos 0}{1^2} - \frac{\cos 0}{2^2} + \frac{\cos 0}{3^2} \cdots\right]$$

$$\frac{\pi^2}{3} = 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} \cdots\right]$$

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} \cdots = \frac{\pi^2}{12} \longrightarrow (2)$$

Adding (1) & (2)

$$\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right) + \left(\frac{1}{1^2} - \frac{1}{2^2} + 1/3^2 - 1/4^2 + \cdots\right) = \frac{\pi^2}{6} + \frac{\pi^2}{12}$$

$$2\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \cdots\right] = \frac{3\pi^2}{12} = \pi^2/4.$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \cdots = \pi^2/8$$

8 find the fourier series for $f(x)$ in the interval $(-\pi, \pi)$ when $f(x) = \begin{cases} \pi + x & -\pi < x < 0. \\ \pi - x & 0 < x < \pi. \end{cases}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx.$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx.$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0} (\pi + x)\,dx + \int_{0}^{\pi} (\pi - x)\,dx\right.$$

$$= \frac{1}{\pi}\left\{\left.\pi x + \frac{x^2}{2}\right]_{-\pi}^{0} + \left.\pi x - \frac{x^2}{2}\right]_{0}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left[+\pi^2 - \frac{\pi^2}{2} + \pi^2 - \frac{\pi^2}{2}\right\} = 1/\pi \cdot \pi^2 \cdot = \pi$$

$$\boxed{a_0 = \pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx.$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0}(\pi+x)\cos nx\,dx + \int_{0}^{\pi}(\pi-x)\cos nx\,dx\right.$$

$$= \frac{1}{\pi}\left\{\left[(\pi+x)\frac{\sin nx}{n} - 1\cdot\frac{-\cos nx}{n^2}\right]_{-\pi}^{0} + \left[(\pi-x)\frac{\sin nx}{n} - -1\times\frac{-\cos nx}{n^2}\right]_{0}^{\pi}\right.$$

$$= \frac{1}{\pi}\left\{\left[\frac{1}{n^2} - \left(\frac{(-1)^n}{n^2}\right)\right] + \left[-\frac{(-1)^n}{n^2} + \frac{1}{n^2}\right]\right.$$

$$= \frac{1}{\pi}\left\{\frac{2}{n^2} - \frac{2(-1)^n}{n^2}\right].$$

$$\boxed{a_n = \frac{2}{\pi n^2}\left[1-(-1)^n\right]}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx.$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(\pi+x)\sin nx\,dx + \int_{0}^{\pi}(\pi-x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[(\pi+x)\times\frac{-\cos nx}{n} - 1\cdot\frac{-\sin nx}{n^2}\right]_{-\pi}^{0} + \left[(\pi-x)\times\frac{-\cos nx}{n} - -1\times\frac{\sin nx}{n^2}\right]_{0}^{\pi}$$

$$= \frac{1}{\pi}\left[\left(-\frac{\pi(1)}{n}\right) + \left(0 - \pi\times -\frac{1}{n}\right)\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n} + \pi/n\right] = 0.$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx.$$

$$= \pi/2 + \sum_{n=1}^{\infty}\frac{2}{\pi n^2}\left(1-(-1)^n\right]\cos nx$$

$$= \frac{\pi}{2} + \frac{2}{\pi}\left[\frac{2}{1^2}\cos x + \frac{2}{3^2}\cos 3x + \frac{2}{5^2}\cos 5x \dots\right]$$

$$\therefore f(x) = \frac{\pi}{2} + \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\right\}$$

Change of Interval (Fourier series of Arbitrary periodic functions).

Suppose $f(x)$ is of length $2l$. in the interval $-l \le x \le l$. Then the Fourier Series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right) \text{ where}$$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx.$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x) \cos\frac{n\pi x}{l}\,dx \qquad n=1,2,3\ldots$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x) \sin\frac{n\pi x}{l}\,dx \qquad n=1,2,3\ldots$$

(length of interval $-2l$. If x is point of discontinuity then $f(x) = \frac{1}{2}\left[f(x^+) + f(x^-)\right]$

If the interval $0 \le x \le 2l$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx. \quad a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx.$$

Problems

1) Find the Fourier Series of $f(x) = x - x^2$ in the interval $-1 \le x \le 1$

Ans: Length of interval $1-(-1) = 2$

$2l = 2 \quad l = 1$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos n\pi x + b_n \sin n\pi x\right)$$

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$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx = \frac{1}{1}\int_{-1}^{1}(x-x^2)\,dx$$

$$= \left.\frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{1} = \left[\frac{1}{2} - \left(\frac{1}{3}\right)\right] - \left(\frac{1}{2} + \frac{1}{3}\right)$$

$$\boxed{a_0 = -2/3} \qquad = \underline{-2/3}.$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx.$$

$$= \int_{-1}^{1}(x-x^2)\cos n\pi x\,dx.$$

$$= \left.(x-x^2)\frac{\sin n\pi x}{n\pi} - (1-2x)\times\frac{-\cos n\pi x}{(n\pi)^2} + -2x - \frac{\sin n\pi x}{(n\pi)^3}\right]_{-1}^{1}$$

$$= +\,-1\frac{\cos n\pi}{(n\pi)^2} - \left(0 + \left(3\frac{\cos n\pi}{(n\pi)^2}\right)\right)$$

$$\boxed{a_n = -\frac{4(-1)^n}{n^2\pi^2}}$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \int_{-1}^{1}(x-x^2)\sin n\pi x\,dx.$$

$$= (x-x^2)\times\frac{-\cos n\pi x}{n\pi} - (1-2x)\times\frac{-\sin n\pi x}{(n\pi)^2}.$$

$$\left. + (-2)\frac{\cos n\pi x}{(n\pi)^3}\right]_{-1}^{1}$$

$$= \frac{-2\cos n\pi}{(n\pi)^3} - \left(2\frac{\cos n\pi}{n\pi} - 2\frac{\cos n\pi}{(n\pi)^3}\right)$$

$$\boxed{b_n = -\frac{2(-1)^n}{n\pi}}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + b_n \sin n\pi x .$$

$$= \frac{-2}{3\times 2} + \sum_{n=1}^{\infty} \frac{-4(-1)^n}{n^2\pi^2} \cos n\pi x + \frac{-2(-1)^n}{n\pi} \sin n\pi x]$$

$$x - x^2 = -\frac{1}{3} + \frac{4}{\pi^2}\left[\frac{1}{1^2} \cos \pi x - \frac{1}{2^2} \cos 2\pi x + \frac{1}{3^2} \cos 3\pi x - \dots \right]$$

$$+ \frac{2}{\pi}\left[\frac{\sin \pi x}{1} - \frac{\sin 2\pi x}{2} + \frac{\sin 3\pi x}{3} - \dots \right]$$

Note: If the interval $-2 \le x \le 2$ then length of the interval is $2-(-2)=4$ i.e. $2l = 4$, $\underline{l=2}$

If the interval $0 \le x \le 4$ then length of interval is $4-0=4$, $2l=4$, $\underline{l=2}$.

Q) Find the Fourier series of the function.

$$f(x) = \begin{cases} 0 & -2 \le x \le -1 \\ k & -1 \le x < 1 \\ 0 & 1 \le x \le 2 \end{cases}$$

A: Here the interval $-2 \le x \le 2$.

Length of interval $2l = 4$, $\underline{l=2}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{2} + b_n \sin \frac{n\pi x}{2}.$$

$$a_0 = \frac{1}{l} \int_{-l}^{l} f(x)\, dx .$$

$$= \frac{1}{2} \int_{-2}^{2} f(x)\, dx .$$

$$= \frac{1}{2}\left[\int_{-2}^{-1} 0\, dx + \int_{-1}^{1} k\, dx + \int_{1}^{2} 0\, dx \right]$$

$$= \frac{1}{2} [kx]_{-1}^{1} = \frac{1}{2}(k - -k) = \frac{2k}{2} = k \quad \boxed{a_0 = k}$$

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l}\, dx .$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx.$$

$$= \frac{1}{2}\left[\int_{-2}^{-1} 0 \cos\frac{n\pi x}{2}\,dx + \int_{-1}^{1} k\cos\frac{n\pi x}{2}\,dx + \int_{1}^{2} 0\cos\frac{n\pi x}{2}\right]$$

$$= \frac{1}{2}\left[\int_{-1}^{1} k\cos\frac{n\pi x}{2}\,dx\right]$$

$$= \frac{k}{2}\left[\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right]_{-1}^{1}.$$

$$= \frac{k}{2}\left[\frac{\sin\frac{n\pi}{2}}{\frac{n\pi}{2}} - \frac{\sin\left(-\frac{n\pi}{2}\right)}{n\pi/2}\right]$$

$$= \frac{k}{2}\; 2\,\frac{\sin\frac{n\pi}{2}}{n\pi/2}$$

$$\boxed{a_n = \frac{2k}{n\pi}\sin n\pi/2.}$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}\,dx$$

$$b_n = \frac{1}{2}.\left[\int_{-1}^{1} k\sin\frac{n\pi x}{2}\,dx\right]$$

$$= \frac{k}{2}\times\left[\frac{-\cos\frac{n\pi x}{2}}{n\pi/2}\right]_{-1}^{1} = -\frac{k}{2}\left[\frac{\cos\frac{n\pi}{2}}{\frac{n\pi}{2}} - \frac{\cos\left(-\frac{n\pi}{2}\right)}{\frac{n\pi}{2}}\right]$$

$$\boxed{b_n = 0}.$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{2}\right) + b_n\sin\frac{n\pi x}{2}$$

$$= \frac{k}{2} + \sum_{n=1}^{\infty} \frac{2k}{n\pi}\sin\frac{n\pi}{2}\cdot\cos\left(\frac{n\pi x}{2}\right).$$

$$= \frac{k}{2} + \frac{2k}{\pi}\left[\frac{\sin\frac{\pi}{2}}{1}\cos\frac{\pi x}{2} + \frac{\sin\frac{2\pi}{2}}{2}\cos\frac{2\pi x}{2} + \frac{\sin\frac{3\pi}{2}}{3}\cos\frac{3\pi x}{2}\right.$$

$$= \frac{k}{2} + \frac{2k}{\pi}\left[\cos\frac{\pi x}{2} - \frac{\cos\frac{3\pi x}{2}}{3} + \frac{\cos\frac{5\pi x}{2}}{5}\cdots\right]$$

H.W

Find the Fourier series for the given function in the interval $0<x<2\pi$

1) $f(x) = e^{-x}$ in the interval $0<x<2\pi$

Ans: $f(x) = \frac{1-e^{-2\pi}}{\pi}\left[\frac{1}{2}+\left(\frac{1}{2}\cos x+\frac{1}{5}\cos 2x+\frac{1}{10}\cos 3x+\cdots\right)+\left(\frac{1}{2}\sin x+\frac{2}{5}\sin 2x+\frac{3}{10}\sin 3x+\cdots\right)\right]$

2). $f(x) = \left(\frac{\pi-x}{2}\right)^2$ in the interval $0<x<2\pi$.

Deduce that $\frac{1}{1^2}+\frac{1}{2^2}+\cdots = \pi^2/6$.

Ans. $f(x) = \frac{\pi^2}{12}+\sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$

3) $f(x) = \begin{cases} 0 & -\pi\le x\le 0 \\ \sin x & 0\le x\le\pi \end{cases}$ Deduce that $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\cdots = 1/2$.

Hint: (deduction put $x=0$

$f(0) = \frac{1}{2}[f(0^+)+f(0^-)]$)

Ans $f(x) = \frac{1}{\pi}+\frac{1}{2}\sin x-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{4n^2-1}$

(4) $f(x) = \begin{cases} -\pi & -\pi<x<0 \\ x & 0<x<\pi \end{cases}$ Deduce that $\frac{1}{1^2}+\frac{1}{3^2}+\cdots = \pi^2/8$.

[Hint: For deduction put $x=0$ in the expansion of $f(x)$]

Ans: $f(x) = -\pi/4-\frac{2}{\pi}\left(\cos x+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\cdots\right)+\left(3\sin x-\frac{\sin 2x}{2}+\sin 3x-\frac{\sin 4x}{4}\cdots\right)$

put $x=0$

$f(0) = \frac{1}{2}[f(0^+)+f(0^-)]$

$= \frac{1}{2}[0+-\pi] = \frac{-\pi}{2}$

$-\frac{\pi}{2} = -\frac{\pi}{4}-\frac{2}{\pi}\left[1+\frac{1}{3^2}+\frac{1}{6^2}+\cdots\right]+0$

$\frac{2}{\pi}\left[1+\frac{1}{3^2}+\frac{1}{5^2}\cdots\right] = \frac{\pi}{2}-\pi/4 = \pi/4 \Rightarrow \left(1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots = \frac{\pi^2}{8}\right.$

$a_n = \frac{1}{l}\int_{-l} f(x)\cos\frac{n\pi x}{l}\,dx$

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Suppose $f(x)$ is defined in the interval $0 \le x \le \pi$ then it has the half range cosine series expansion given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

Similarly the function has the half range sine series expansion given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad \text{Where.}$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx.$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx \qquad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx.$$

Suppose $f(x)$ is defined in the interval $0 \le x \le l$, then the function has the half - range cosine series expansion given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}. \quad \text{where } a_0 = \frac{2}{l}\int_0^{l} f(x)\,dx.$$

$$a_n = \frac{2}{l}\int_0^{l} f(x)\cos\frac{n\pi x}{l}\,dx \quad n = 1, 2, \dots$$

Half range. Sine series expansion is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad \text{Where } b_n = \frac{2}{l}\int_0^{l} f(x)\sin\frac{n\pi x}{l}\,dx.$$

$n = 1, 2, \dots$

Pbms:

1. Expand $\pi - x$ in a half-range sine series in the interval $0 \le x \le \pi$ up-to the first three terms.

A: Half - range sine series expansion of $f(x)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \quad \text{Where } b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi}(\pi - x)\sin nx\,dx.$$

$$= \frac{2}{\pi}\left[(\pi - x)\frac{-\cos nx}{n} - {-1}\times\frac{-\sin nx}{n^2}\right]_0^{\pi}$$

$$\frac{2}{\pi}\left[\frac{\pi}{n}\right] = \frac{2}{n}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx = \sum_{n=1}^{\infty} \frac{2}{n} \sin nx.$$

$$= 2\left[\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \dots\right]$$

2. Find the half range cosine series for the function $f(x) = x^2$ in the range $0 \le x \le \pi$

A: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx. = 2/\pi \int_0^{\pi} x^2\,dx = \frac{2}{\pi}\left(\frac{x^3}{3}\right)_0^{\pi}$$

$$= \frac{2}{\pi}\,\pi^3/3 = \frac{2\pi^2}{3} \qquad \boxed{a_0 = \frac{2\pi^2}{3}}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2 \cdot \cos nx\,dx$$

$$= \frac{2}{\pi}\left[x^2 \times \frac{\sin nx}{n} - 2x \times -\frac{\cos nx}{n^2} + 2 \times -\frac{\sin nx}{n^3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[2\pi \frac{\cos n\pi}{n^2}\right] = \frac{4}{n^2}(-1)^n \qquad \boxed{a_n = \frac{4(-1)^n}{n^2}}$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$= \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx.$$

$$= \frac{\pi^2}{3} + 4\left[-\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} \dots\right]$$

$$= \frac{\pi^2}{3} - 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} \dots\right]$$

8. Obtain half range cosine series expansion of the function $f(x) = \begin{cases} kx & 0 \le x \le l/2 \\ k(l-x) & l/2 \le x \le l \end{cases}$

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \dots = \pi^2/8$

$\rightarrow$ $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{l}\left[\int_0^{l/2} kx\,dx + \int_{l/2}^{l} k(l-x)\,dx\right]$$

$$= \frac{2}{l}\left[\left(k\frac{x^2}{2}\right)_0^{l/2} + k\left[lx - x^2/2\right]_{l/2}^{l}\right]$$

$$= \frac{2}{l}\left[\frac{k}{2}\frac{l^2}{4} + k\left(l^2 - l^2/2 - \left(\frac{l^2}{2} - l^2/8\right)\right)\right]$$

$$= \frac{2}{l}\left[\frac{kl^2}{8} + kl^2 - \frac{kl^2}{2} - kl^2/2 + kl^2/8\right] = \frac{kl}{2}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx.$$

$$= \frac{2}{l}\left\{\int_0^{l/2} kx \cos\frac{n\pi x}{l}\cdot dx + \int_{l/2}^{l} k(l-x)\cos\frac{n\pi x}{l}\,dx\right.$$

$$= \frac{2k}{l}\left\{\left[x\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}} - \frac{-\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right]_0^{l/2} + \left[(l-x)\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}} - {-1}\times\frac{-\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right]_{l/2}^{l}\right\}$$

$$= \frac{2k}{l}\left\{\left[l/2\frac{\sin\frac{n\pi}{2}}{\frac{n\pi}{l}} + \frac{\cos\frac{n\pi}{2}}{\left(\frac{n\pi}{l}\right)^2} - \left(0 + \frac{1}{\left(\frac{n\pi}{l}\right)^2}\right)\right]\right.$$

$$+ \left[\left[0 - \cos n\pi\right] - \left[l/2\,\sin\frac{n\pi}{2} - \cos\frac{n\pi}{2}\right]\right]$$

$$\frac{2k}{l}\left\{\frac{l}{2}\frac{\sin\frac{n\pi}{2}}{\left(\frac{n\pi}{l}\right)}+\frac{\cos n\pi/2}{\left(\frac{n\pi}{l}\right)^2}-\frac{1}{\left(\frac{n\pi}{l}\right)^2}\right.$$

$$\frac{2k}{l}\left\{\frac{l^2}{2n\pi}\sin\frac{n\pi}{2}+\frac{l^2}{n^2\pi^2}\cos\frac{n\pi}{2}-\frac{l^2}{n^2\pi^2}-\frac{l^2(-1)^n}{n^2\pi^2}+\right.$$

$$\left.\frac{l^2}{n^2\pi^2}\cos\frac{n\pi}{2}-\frac{l^2}{2n\pi}\sin n\pi/2\right\}$$

$$=\frac{2k}{l}\cdot\frac{l^2}{n^2\pi^2}\left\{2\cos\frac{n\pi}{2}-1-(-1)^n\right\}$$

$$=\frac{2kl}{n^2\pi^2}\left[2\cos\frac{n\pi}{2}-1-(-1)^n\right]$$

When n odd $\cos n\pi/2=0$, $(-1)^n=-1$

$a_1=0$ $a_3=0$, $a_5=0$, ...

When n is even $a_2=\frac{2kl}{2^2\pi^2}\left[2\cos\pi-1-1\right]=-\frac{8kl}{\pi^2\cdot 2^2}$

$$a_4=\frac{2kl}{4^2\pi^2}\left[2\cos 2\pi-1-1\right]=0$$

$$a_6=\frac{2kl}{6^2\pi^2}\left[2\cos 6\pi-1-1\right]=-\frac{8kl}{\pi^2\cdot 6^2}$$ & so on

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l}$$

$$=\frac{kl}{4}+\left[-\frac{8kl}{\pi^2 2^2}\cos\frac{2\pi x}{l}-\frac{8kl}{\pi^2\cdot 6^2}\cos\frac{6\pi x}{l}\cdots\right]$$

$$=\frac{kl}{4}-\frac{8kl}{\pi^2}\left[\frac{\cos\frac{2\pi x}{l}}{2^2}+\frac{\cos\frac{6\pi x}{l}}{6^2}+\cdots\right]$$

Put $x=l$. $f(l)=0$

$$\therefore 0=\frac{kl}{4}-\frac{8kl}{\pi^2}\left[\frac{\cos 2\pi}{2^2}+\frac{\cos 6\pi}{6^2}+\cdots\right]$$

$$\frac{kl}{4}=\frac{8kl}{\pi^2}\left[\frac{1}{2^2}+\frac{1}{6^2}+\cdots\right]\Rightarrow\frac{1}{2^2}+\frac{1}{6^2}+\cdots=\frac{\pi^2}{32}$$

$$\frac{1}{2^2}\left[\frac{1}{1^2}+\frac{1}{3^2}+\cdots\right] = \frac{\pi^2}{32}$$

$$\frac{1}{1^2}+\frac{1}{3^2}+\cdots = \frac{\pi^2}{8}.$$

Q. Find half-range Sine Series for

$$f(x) = \begin{cases} \frac{1}{4}-x, & 0<x<1/2 \\ x-3/4, & 1/2<x<1. \end{cases}$$

$l=1$

$$f(x) = \cancel{a_0} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n c$$

$$= \sum_{n=1}^{\infty} b_n \sin n\pi x.$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{1}\left[\int_0^{1/2}\left(\frac{1}{4}-x\right)\sin n\pi x\,dx + \int_{1/2}^{1}\left(x-\frac{3}{4}\right)\sin n\pi x\,dx\right.$$

$$=2\left\{\left[\left(\frac{1}{4}-x\right)\frac{-\cos n\pi x}{n\pi} - (-1)\times\frac{-\sin n\pi x}{(n\pi)^2}\right]_0^{1/2} + \left[\left(x-3/4\right)\times\frac{-\cos n\pi x}{n\pi}\right.\right.$$

$$\left.\left. - \frac{-\sin n\pi x}{(n\pi)^2}\right]_{1/2}^{1}\right\},$$

$$= 2\left\{\left[+\frac{1}{4}\cancel{\frac{\cos\frac{n\pi}{2}}{n\pi}} - \frac{\sin\frac{n\pi}{2}}{(n\pi)^2}\right] - \left(-\frac{1}{4}\frac{\cos 0}{n\pi}\right) + \right.$$

$$\left.\left[-\frac{1}{4}\frac{\cos n\pi}{n\pi}\right] - \left[\frac{1}{4}\cancel{\frac{\cos\frac{n\pi}{2}}{n\pi}} + \frac{\sin\frac{n\pi}{2}}{(n\pi)^2}\right]\right\}$$

$$=2\left\{\frac{1}{4n\pi} - \frac{2\sin\frac{n\pi}{2}}{n^2\pi^2} - \frac{1}{4n\pi}(-1)^n\right\}$$

$$=2\left\{\frac{1-(-1)^n}{4n\pi} - \frac{2\sin\frac{n\pi}{2}}{n^2\pi^2}\right\}$$

When n even $\quad \frac{1-(-1)^n}{4n\pi} = 0 \qquad \sin\frac{n\pi}{2} = 0$

$\therefore b_n = 0$

$b_1 = 2\left[\frac{1}{2\pi} - \frac{2}{\pi^2}\right] = \frac{1}{\pi} - \frac{4}{\pi^2}$

$b_3 = 2\left[\frac{1}{6\pi} + \frac{2}{3^2\pi^2}\right] = \frac{1}{3\pi} + \frac{4}{\pi^2 . 3^2}$

$b_5 = 2\left[\frac{1}{10\pi} - \frac{2}{5^2\pi^2}\right] = \frac{1}{5\pi} - \frac{4}{5^2\pi^2}$

$\therefore f(x) = \left(\frac{1}{\pi} - \frac{4}{\pi^2}\right)\sin\pi x + \left(\frac{1}{3\pi} + \frac{4}{3^2\pi^2}\right)\sin 3\pi x + \left[\frac{1}{5\pi} - \frac{4}{5^2\pi^2}\right]\sin 5\pi x + \dots$

FOURIER SERIES

Full Range

$-\pi < x < \pi$

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

$0 < x < 2\pi$

$a_0 = \frac{1}{\pi}\int_{0}^{2\pi} f(x)\,dx$

$a_n = \frac{1}{\pi}\int_{0}^{2\pi} f(x)\cos nx\,dx$

$b_n = \frac{1}{\pi}\int_{0}^{2\pi} f(x)\sin nx\,dx$

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + b_n\sin nx$

$-l < x < l$

$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$

$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$

$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$

$0 < x < 2l$

$a_0 = \frac{1}{l}\int_{0}^{2l} f(x)\,dx$

$a_n = \frac{1}{l}\int_{0}^{2l} f(x)\cos\frac{n\pi x}{l}\,dx$

$b_n = \frac{1}{l}\int_{0}^{2l} f(x)\sin\frac{n\pi x}{l}\,dx$

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}$.

Half Range

$0 < x < \pi$.

$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$

$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$

$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + b_n\sin nx$

$0 < x < l$.

$a_0 = 2/l\int_0^{l} f(x)\,dx$

$a_n = 2/l\int_0^{l} f(x)\cos\frac{n\pi x}{l}\,dx$

$b_n = 2/l\int_0^{l} f(x)\sin\frac{n\pi x}{l}\,dx$

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}$

Parseval's thm

If the Fourier Series over an interval $c < x < c+2l$ is given as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right]$$

then $$\frac{1}{2l}\int_c^{c+2l} [f(x)]^2\, dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}\left[a_n^2 + b_n^2\right]$$

Example 1. *Find the Fourier sine series for unity in $0 < x < \pi$ and hence show that*

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots\ldots = \frac{\pi^2}{8}.$$

Sol. We require half-range Fourier sine series for 1 in $(0, \pi)$

Let $$1 = \sum_{n=1}^{\infty} b_n \sin nx$$

Then $$b_n = \frac{2}{\pi}\int_0^{\pi} (1) \sin nx\, dx = \frac{2}{\pi}\left[-\frac{\cos nx}{n}\right]_0^{\pi} = -\frac{2}{n\pi}(\cos n\pi - 1)$$

$$= \frac{2}{n\pi}[1 - (-1)^n] \qquad [\because \cos n\pi = (-1)^n]$$

Now $b_n = 0$ when n is even ; and $b_n = \dfrac{4}{n\pi}$ when n is odd.

Substituting in (1), we get

$$\therefore\quad 1 = \sum_{m=1}^{\infty} \frac{4}{(2m-1)\pi} \sin(2m-1)x \quad \text{or} \quad 1 = \frac{4}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\ldots\right)$$

Now from Parseval's theorem on Fourier constants

$$\int_c^{c+2l} [f(x)]^2\, dx = 2l\left[\frac{a_0^2}{4} + \frac{l}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right]$$

Applying (3) to half-range sine series for 1 in $(0, \pi)$

$$c = 0,\ 2l = \pi,\ f(x) = 1,\ a_0 = 0,\ a_n = 0, \text{ and } b_n = \frac{4}{(2m-1)\pi},\ m = 1, 2, \ldots\ldots$$

We get, $$\int_0^{\pi} (1)^2\, dx = \pi \cdot \frac{1}{2} \sum_{m=1}^{\infty} \frac{16}{(2m-1)^2} \cdot \pi^2$$

$$\left[x\right]_0^{\pi} = \frac{8}{\pi}\left\{\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\right\} \quad \text{or} \quad \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots$$

Hence the result.

Example 2. *Find Fourier series of x^2 in $(-\pi, \pi)$. Use Parseval's identity to prove that*

$$\frac{\pi^4}{90} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \ldots\ldots$$

Sol. The Fourier series of x^2 in $(-\pi, \pi)$ are

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx \qquad \ldots(1)$$

Here $$a_0 = \frac{2\pi^2}{3},\ a_n = \frac{4(-1)^n}{n^2},\ b_n = 0,\ f(x) = x^2$$

Now using Parseval's identity to (1)

$$\int_{-\pi}^{\pi} (x^2)^2\, dx = 2\pi\left[\frac{\pi^4}{9} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{16}{n^4}\right]$$

$$\left[\frac{x^5}{5}\right]_{-\pi}^{\pi} = \frac{2\pi^5}{9} + \pi \sum_{n=1}^{\infty} \frac{16}{n^4} \quad \text{or} \quad \frac{2\pi^5}{5} - \frac{2\pi^5}{9} = \pi \sum_{n=1}^{\infty} \frac{16}{n^4}$$

or $$\frac{\pi^4}{90} = \sum_{n=1}^{\infty} \frac{1}{n^4} \quad \text{or} \quad 1 + \frac{1}{2^4} + \frac{1}{3^4} + \ldots\ldots = \frac{\pi^4}{90}.$$

Taylor and Maclaurins Series

If f has derivatives of all order at x_0, then (33)

we call the series

$$\sum_{k=0}^{\infty} \frac{f^k(x_0)}{k!}(x-x_0)^k = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots + \frac{f^k(x_0)}{k!}(x-x_0)^k + \dots$$

the taylor series for f about $x = x_0$. In the special case where $x_0 = 0$, this series become.

$$\sum_{k=0}^{\infty} \frac{f^k(0)}{k!}(x^k) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^k(0)}{k!}(x)^k + \dots$$

in which case we call it the maclaurins series for f.

$\sum_{k=0}^{\infty} c_k x^k = c_0 + c_1 x + c_2 x^2 \dots + c_k x^k$ is called a power series in x.

1. find the Taylor series expansion for the function e^x about $x = -1$

Soln Here $x_0 = -1$

$\therefore$ The taylor series expansion of the function $f(x)$ about $x = x_0$ is given by

$$f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots + \frac{f^k(x_0)}{k!}(x-x_0)^k + \dots$$

(49)

Here $f(x) = e^x$ $\quad f(x_0) = f(-1) = e^{-1}$

$f'(x) = e^x$ $\quad f'(x_0) = f'(-1) = e^{-1}$

$f''(x) = e^x$ $\quad f''(x_0) = f''(-1) = e^{-1}$

$\cdots$

$f^{k}(x) = e^x$ $\quad f^{k}(x_0) = f^{k}(-1) = e^{-1}$

ॐ Taylor Series expansion of e^x about $x = -1$ given by

$$e^{-1} + \frac{e^{-1}}{1!}(x - -1) + \frac{e^{-1}}{2!}(x - -1)^2 + \cdots + \frac{e^{-1}}{k!} + \cdots$$

$$= e^{-1} + \frac{e^{-1}}{1!}(x+1) + \frac{e^{-1}}{2!}(x+1)^2 + \cdots + \frac{e^{-1}}{k!}$$

$$= \sum_{k=0}^{\infty} \frac{(x+1)^k}{k!\, e}$$

Here $P_0(x) = f(x_0) = e^{-1}$

$P_1(x) = f(x_0) + f'(x_0)(x - x_0) = e^{-1} + e^{-1}(x+1)$

$P_2(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 = e^{-1} + e^{-1}(x+1) + \frac{e^{-1}(x+1)^2}{2!}$

$\vdots$

$$P_n(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \cdots + \frac{f^{n}(x_0)}{n!}(x - x_0)^n$$

$$= e^{-1} + \frac{e^{-1}}{1!}(x+1) + \cdots + \frac{e^{-1}}{n!}(x+1)^n$$

$P_0(x), P_1(x), P_2(x), \cdots P_n(x)$ is called Taylor Polynomial for function $f(x)$.

(50) 08

Find the maclaurin Series for (i) $\sin x$ (ii) $\cos x$ (iii) $\frac{1}{1-x}$

(i) $f(x) = \sin x$. $\quad f(0) = 0$

$f'(x) = \cos x \quad f'(0) = 1$

$f''(x) = -\sin x \quad f''(0) = 0$

$f'''(x) = -\cos x \quad f'''(0) = -1$

(35)

Maclaurin Series for $\sin x$ is given by

$$f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \cdots + \frac{f^k(0)}{k!}(x-0)^k + \cdots$$

$$f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^k(0)}{k!}x^k + \cdots$$

$$0 + \frac{1}{1!}x + 0 + \frac{-1}{3!}x^3 + 0 + \frac{1}{5!}x^5 + \cdots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

$P_0(x) = f(0) = 0$

$P_1(x) = f(0) + f'(0)(x-0)$

$P_2(x) = f(0) + f'(0) + f''(0) = 0+1+0 = 1$

$P_0(x), P_1(x), P_2(x) \cdots$

$P_n(x) = f(0) + f'(0) + \cdots + f^n(0)$

$= 0+1+0+\cdots+$

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(5)

$P_0(0) = f(0) = 0$

$P_1(0) = f(0) + f'(0)(x) = 0 + x$

$P_n(0) = f(0) + f'(0)x + \dots + f^n(0)x^n$

$= 0 + x + 0 - \frac{x^3}{3!} \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$P_0(0), P_1(0), P_2(0) \dots$ $P_n(0)$ is called a Mac[laurin]

Polynomial.

(2) $f(x) = \cos x$ $\quad f(0) = 1$ $\quad f^{IV}(x) = \cos x \quad f^{IV}(0)$:

$f'(x) = -\sin x$ $\quad f'(0) = 0$

$f''(x) = -\cos x$ $\quad f''(0) = -1$

$f'''(x) = \sin x$ $\quad f'''(0) = 0$

for $\cos x$ is given by maclurin Series expansion

$f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^k(0)}{k!}$

$1 + 0 + -\frac{x^2}{2!} + 0 + \frac{x^4}{4!} + 0 - \dots + (-1)^k$

$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + \frac{(-1)^k x^{2k}}{2k!}$

(3) $f(x) = \frac{1}{1-x}$

$f(0) = 1$

$f'(x) = \frac{-1}{(1-x)^2} \times (-1) = \frac{1}{(1-x)^2}$ $\quad f'(0) = 1$

$f''(x) = \frac{-2}{(1-x)^3} \times (-1) =$ $\quad f''(0) = 2 = 2!$

$f'''(x) = \frac{3\times 2}{(1-x)^4} = \frac{6}{(1-x)^4}$ $\quad f'''(0) = 6 = 3!$

(24)

- $f^{k}(x) = \frac{k!}{(1-x)^{k+1}}$ $f^{k}(0) = k!$

Maclaurin Series expansion of $\frac{1}{1-x}$ is given by

$f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{k}(0)}{k!}x^k \cdots$

$1 + \frac{1}{1!}x + \frac{2!}{2!}x^2 + \frac{3!}{3!}x^3 + \cdots + \frac{k!}{k!}x^k \cdots$

$= 1 + \frac{x}{1!} + x^2 + x^3 + \cdots + x^k \cdots$

$= \sum_{k=0}^{\infty} x^k$

* find the Taylor Series expansion of $\frac{1}{x}$ about $x = -1$

$f(x) = \frac{1}{x}$ $\cancel{f(0)}$ $f(x_0) = f(-1) = -1$

$f'(x) = \frac{-1}{x^2}$ $f'(x_0) = f'(-1) = \frac{-1}{1} = -1$

$f''(x) = \frac{-2 \times (-1)}{x^3} = \frac{2}{x^3}$ $f''(x_0) = \frac{2}{-1} = -2$

$f'''(x) = \frac{-6}{x^4}$ $f'''(x_0) = f'''(-1) = -6$

$f(-1) + \frac{f'(-1)}{1!}(x+1) + \frac{f''(-1)}{2!}(x+1)^2 + \cdots + \frac{f^{k}(-1)}{k!}(x+1)^k + \cdots$

$= -1 + \frac{-1}{1!}(x+1) + \frac{-2}{2!}(x+1)^2 + \cdots - (x+1)^k \cdots$

$= -1 - (x+1) - (x+1)^2 + \cdots$

$= -\left[1 + (x+1) + (x+1)^2 + \cdots + (x+1)^k + \cdots \right]$

$= -\sum_{k=0}^{\infty} (x+1)^k$

* $f(x) = \ln x$, $x_0 = e$

$f(x) = \ln x \qquad f(x_0) = f(e) = \ln e = 1$

$f'(x) = \frac{1}{x} \qquad f'(x_0) = f'(e) = \frac{1}{e}$

$f''(x) = \frac{-1}{x^2} \qquad f''(x_0) = f''(e) = \frac{-1}{e^2}$

$f'''(x) = \frac{-2x^{-1}}{x^3} = \frac{2}{x^3} \qquad f'''(x_0) = f'''(e) = \frac{2}{e^3}$

Taylor series expansion of $\ln x$ is given by

$$f(e) + \frac{f'(e)(x-e)}{1!} + \frac{f''(e)(x-e)^2}{2!} + \frac{f'''(e)(x-e)^3}{3!} + \cdots$$

$$= 1 + \frac{1}{1!}\frac{1}{e}(x-e) + \frac{1}{2!}\frac{-1}{e^2}(x-e)^2 + \frac{1}{3!}\frac{2}{e^3}(x-e)^3 + \cdots$$

$$= 1 + \left[\frac{1}{1!}\frac{(x-e)}{e} - \frac{(x-e)^2}{2\cdot e^2} + \frac{(x-e)^3}{3e^3} + \cdots (-1)^{k+1}\frac{(x-e)}{e^k}\right.$$

$$= 1 + \sum_{k=0}^{\infty} (-1)^{k+1} \frac{(x-e)^k}{k e^k}$$

H.W $f(x) = \sin \pi x$ about $x = 0$

$f(x) = x e^x$ about $x = 0$

→ $f(0) = 0$

$f'(x) = x e^x + e^x \qquad f'(0) = 1$

$f''(x) = x e^x + 2e^x \qquad f''(0) = 2$

$$x e^x = 0 + \frac{1}{1!}x + \frac{2}{2!}x^2 + \frac{3}{3!}x^3 + \cdots + \frac{k}{k!}x^k + \cdots$$

$$= x + \frac{x^2}{1!} + \frac{x^3}{2!} + \cdots + \frac{x^n}{(n-1)!}$$

■ BINOMIAL SERIES

If m is a real number, then the Maclaurin series for $(1+x)^m$ is called the ***binomial series***; it is given by

$$1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots + \frac{m(m-1)\cdots(m-k+1)}{k!}x^k + \cdots$$

In the case where m is a nonnegative integer, the function $f(x) = (1+x)^m$ is a polynomial of degree m, so

$$f^{(m+1)}(0) = f^{(m+2)}(0) = f^{(m+3)}(0) = \cdots = 0$$

and the binomial series reduces to the familiar binomial expansion

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots + x^m$$

which is valid for $-\infty < x < +\infty$.

It can be proved that if m is not a nonnegative integer, then the binomial series converges to $(1+x)^m$ if $|x| < 1$. Thus, for such values of x

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \cdots + \frac{m(m-1)\cdots(m-k+1)}{k!}x^k + \cdots \qquad (17)$$

or in sigma notation,

$$(1+x)^m = 1 + \sum_{k=1}^{\infty} \frac{m(m-1)\cdots(m-k+1)}{k!}x^k \quad \text{if } |x| < 1 \qquad (18)$$

▶ **Example 4** Find binomial series for

(a) $\dfrac{1}{(1+x)^2}$ (b) $\dfrac{1}{\sqrt{1+x}}$

Solution (a). Since the general term of the binomial series is complicated, you may find it helpful to write out some of the beginning terms of the series, as in Formula (17), to see developing patterns. Substituting $m = -2$ in this formula yields

$$\begin{aligned}
\frac{1}{(1+x)^2} = (1+x)^{-2} &= 1 + (-2)x + \frac{(-2)(-3)}{2!}x^2 \\
&\quad + \frac{(-2)(-3)(-4)}{3!}x^3 + \frac{(-2)(-3)(-4)(-5)}{4!}x^4 + \cdots \\
&= 1 - 2x + \frac{3!}{2!}x^2 - \frac{4!}{3!}x^3 + \frac{5!}{4!}x^4 - \cdots \\
&= 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \cdots \\
&= \sum_{k=0}^{\infty} (-1)^k (k+1) x^k
\end{aligned}$$